ELEMENTS OF POINT-SET TOPOLOGY

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Preface

Nowadays one speaks more and more about the specialization in modern science. Even though this statement is valid up to a certain point, one might say that a characteristic of science today is the every time greater interaction among the various disciplines that conform it. Similarly to to what happens in science in general, in each discipline one pursues a broader relationship among the different fields that conform it. In mathematics, for instance, one expects from a differential geometer or from a function theorist a much wider common knowledge than the one required one half century ago. This happens because of the ubiquity that some mathematical concepts show more and more. One of these mathematical concepts is that of a topological space, that includes everything related with "nearness", "continuity", "neighborhood", "deformation", et cetera.

Topology has been for many years one of the most important and influential fields in modern mathematics. Its origins date back over some centuries, but it was doubtless Poincaré who imprinted the great impetus that topology gained throughout the twentieth century. There are other great names among those who created point-set topology, whose existence has been justified by the great progress of algebraic topology. On the one hand, the effectiveness of point-set topology, more than due to deep theorems, it rests in the first place on its conceptual simplicity and on its convenient terminology, because in a sense it establishes a link between abstract, not very intuitive problems, and our ability to visualize geometric phenomena in space. This intellectual ability to grasp what is going on in 3-dimensional space, that through topology allows us to delve into mathematical thinking and into the world of abstract objects, is very independent of abstraction and logical thinking. This reinforcement of our mathematical talent is probably the deepest cause of the effectiveness and the simplicity of the topological methods.

As many of the basic mathematical branches, topology has an intricate history. If we mark the start of topology at the point when the conceptual system of point-set topology was established, then we have to refer to Felix Hausdorff's book *Grundzüge der Mengenlehre* (Foundations of Set Theory), Leipzig, 1914, in whose Chapter 7 "Point Sets in General Spaces", he establishes the most important and basic concepts in point-set topology. Already in 1906, in his paper *Sur* *quelques points du calcul fonctionnel* (On some Topics of Functional Calculus), Maurice Fréchet introduced the concept of metric space and he tried to establish the concept of topological space, by giving an axiomatic approach to the concept of convergence. What Fréchet really created were the topological foundations of functional analysis.

But, of course, the history dates further back to the times when the effervescence of geometry was taking place during the nineteenth century. At the beginning of that century there was the classical idea that geometry was the mathematical ambit, where the concepts of the physical space developed. Towards the end of that century, as it was shown by Felix Klein in his *Erlanger Programm* (Erlangen Program. Comparative Considerations about New Geometric Investigations), the projection went much farther than the physical space and it even started to consider such abstract spaces as the n-manifolds, the projective spaces, the Riemann surfaces, or even the function spaces.

Among the decisive works for the emergence of topology, one finds the monumental work of Georg Cantor. He established the bases on which the abstract concept of a topological space is formulated as "a set furnished with a collection of subsets such that..." Indeed, already in 1870, Cantor had shown that if two Fourier series converge pointwise and have the same limit, then they must have the same coefficients. Cantor himself improved this result in 1871 by showing that the coincidence of the coefficients can equally be achieved by requiring pointwise convergence or equality of the limits, up to a finite set in the interval $[0, 2\pi]$. In 1872 he analyzed certain infinite subsets, up to which his statement remains valid. It was then, when he introduced his famous *Cantor set*, that being "only" a subset of an interval, it is topologically not only a very interesting object, but of great importance in several branches of mathematics.

The problem of deciding if two spaces are homeomorphic or not is no doubt the central problem in topology. It was not until the creation of algebraic topology that it was possible to give a reasonable answer to such a problem. Now it is not only because of its conceptual simplicity and its adequate symbology, but thanks to the powerful tool provided by algebra and its most convenient functorial relationship to topology that this effectiveness is achieved.

The analytic description of dynamical systems in classical mechanics represented the first step towards the necessity to create a geometrical language in dimensions higher than the usual ones. Already in the eighteenth century, Lagrange, in his *Mécanique analytique* (Analytical Mechanics) Paris, 1788, had considered the possibility of grasping a fourth dimension. It was Riemann, in his famous *Habilitationsvortrag: Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the Hypotheses underlying Geometry), Göttingen, 1854, who presented the first ideas on the geometry of manifolds.

Lagrange himself, in his *Leçons sur le Calcul des Fonctions* (Lessons on Functional Calculus), Paris, 1806, introduced the concept of perturbation, or homotopy, of curves in variational calculus problems to detect certain minimal curves. What we now know as algebraic topology was probably started with Henri Poincaré's *Analysis Situs*, Paris, 1895, and its five *Compléments* (Complements), Palermo 1899, London 1900, Paris 1902, Paris 1902, and Palermo 1904. In the first, he notices that "*geometry is the art of reasoning well with badly made figures.*" And further says: "*Yes, without doubt, but under one condition. The proportions of the figures might be grossly altered, but their elements must not be interchanged and must preserve their relative situation. In other terms, one does not worry about quantitative properties, but one must respect the qualitative properties. That is to say precisely those, which are the concern of Analysis Situs.*" Indeed, other works of Poincaré contain as much interesting topology as the ones just referred to. This is the case for his memoir on the qualitative theory of differential equations, that includes the famous formula of the Poincaré index. This formula describes in topological terms the famous Euler formula, and constitutes one of the first steps in algebraic topology. In the works mentioned, Poincaré considers already maps on manifolds such as, for instance, vector fields, whose indexes determine the Euler characteristic in his index formula. It is Poincaré too who generalizes the question on the classification of manifolds having in mind the classification of the orientable surfaces considered by Moebius in his *Theorie der elementaren Verwandtschaft* (Theory of elementary relationship), Leipzig, 1863. This classification problem was also solved by Jordan in *Sur la déformation des Surfaces* (On the deformation of surfaces), Paris, 1866, who, by classifying surfaces solved an important homeomorphism problem.

Jordan also studied the homotopy classes of closed paths, that is, the first notions of the fundamental group, inspired by Riemann, who already had analyzed the behavior of integrals of holomorphic differential forms and therewith the concept of homological equivalence between closed paths.

Of course, Cantor, Fréchet, Klein, Hausdorff, Riemann, Jordan, Moebius, and Poincaré are not the only architects of the basic concepts of topology. All this history is in itself the object of another text. Doubtless, the text [10] edited by I.M. James is an excellent reference in that direction.

This book has the purpose of presenting the topics of point-set topology, which from my own point of view, are basic for an undergraduate student, who is interested in this area or affine areas in mathematics.

The design of the text is as follows. We start with a small rather motivating Chapter 1, followed by six substantial chapters, each of which is divided into several sections that are distinguished by their double numbering (1.1, 1.2, 2.1, ...). Definitions, propositions, theorems, remarks, formulas, exercises, etc., are designated with triple numbering (1.1.1, 1.1.2, ...). Exercises are an important part of the text, since many of them are intended to carry the reader further along the lines already developed, in order to prove results that are either important by themselves or relevant for future topics. Most of these are numbered, but occasionally they are identified inside the text by the use of italics (*exercise*).

The book starts considering metric spaces to arrive to the abstract properties of their open sets. They lead us to the abstract concept of a topological space. Then it studies continuity as the fundamental property expected from any function between topological spaces. Further, one studies several conditions which are additional to the very axioms, that guarantee useful and convenient properties of the spaces. Special emphasis is put on compact spaces, because this concept has a special importance in several applications of topology. We finish with the metrizability theorems and the Stone–Čech compactification. Along the book, we remark the universal properties that the many of the given constructions have. Particularly the universal properties that characterize the topological sum and the topological product. We also give the universal properties of the identifications, the Stone–Čech compactification, et cetera. A section is devoted to the important topic on limits and colimits of topological spaces, and one more to the compactly generated spaces, since they play an important role in algebraic topology.

The book is intended to be used in a one-semester course in the higher undergraduate level. To do this, one may read Chapters 1–3. In Chapter 4, one can leave out Section 4.6, which refers to limits and colimits. In Chapter 5, one can omit Section 5.6, which is devoted to nets. In Chapter 6, Section 6.7, which handles compactly generated spaces, can be left out. Finally in Chapter 7, Sections 7.5 and 7.6, which deal with paracompact spaces and with the interrelations among several properties, can be omitted. This shortcut fulfills the purpose of the book of starting with metric spaces and, after adding adequate conditions to general topological spaces, coming back to metrizable spaces. On the other hand, the omitted sections can be left to the students to develop different projects. In particular, Section 6.7, besides the compactly generated spaces, describes briefly the class of k-spaces, which represents a very interesting project for good students.

At this point I want recognize the big impact in this book of all experts that directly or indirectly have influenced my education as a mathematician and as a topologist. At the UNAM, Guillermo Torres and Roberto Vázquez were decisive. Later on, in my doctoral studies in Heidelberg, Germany, I had the privilege of receiving directly the teaching of Albrecht Dold and Dieter Puppe. Indirectly, I was influenced by some German topology books, among which I owe a mention to that of Klaus Jänich [11] –whose effects are clearly reflected, mainly in this

preface–. Horst Schubert's influence [16] is clear all through the book.

Mexico City, Mexico Carlos Prieto
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Introduction

The object of this book is to study topology. But, what is topology? This is not an easy question to answer. Trying to define a branch of mathematics in a concise sentence is complicated. However, as an approximation, we can say that topology is the branch of mathematics that studies continuous deformations of geometric objects. One of the purposes of topology is to classify objects, or at least, to give methods in order to distinguish between objects that are not homeomorphic. Namely to decide between objects that cannot be obtained from each other through a continuous deformation. Topology also provides techniques to study topological structures in objects that arise in very different fields of mathematics. Those concepts such as "deformation, "continuity" and "homeomorphism" are fundamental and will be precisely defined throughout the text. However, although we have not defined these concepts yet, we introduce in what follows several examples, that intuitively illustrate those concepts.

Figure 0.1 shows three topological spaces, namely a spherical surface with its poles deleted, another sphere with its polar caps removed (including the polar circles), and a cylinder with its top and bottom removed (including the edges). These three objects can clearly be deformed one to the other. Topology would not distinguish them.

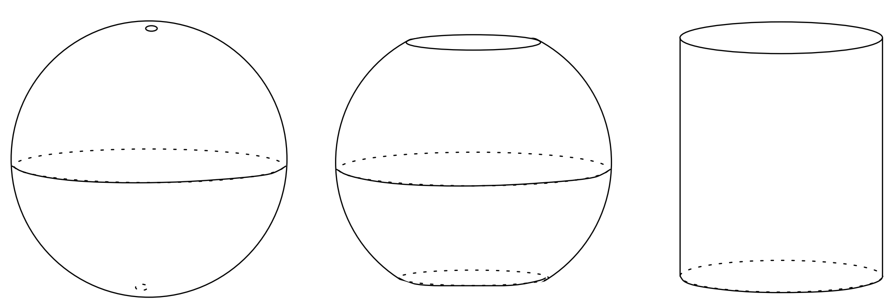


Figure 0.1 A sphere with no poles, a sphere with no polar caps and circles, and a cylinder with no edges

Figure 0.2 shows the surface of a torus (a "doughnut") and a spherical surface with a handle attached. Each one of these objects is clearly a deformation of the other. However, it is intuitively clear that the topological object shown in three forms in Figure 0.1, i.e. the 2-sphere with no poles and the other two, cannot be deformed to the topological object shown in two forms in Figure 0.2.

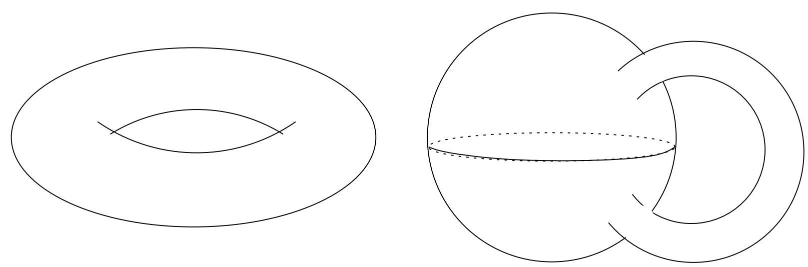


Figure 0.2 A torus and a sphere with one handle

Being a little more precise, we shall say that two objects (topological spaces) will be *homeomorphic* when there exists a one-to-one correspondence, which maps points that are *close* in one of the objects to points that are close in the other. We can add to the list of those spaces, mentioned above, which are not homeomorphic to each other, the following examples:

(a) Take $N = \{0, 1, \ldots, n-1\}$, $M = \{0, 1, \ldots, m-1\}$, $n < m$. Considered as topological spaces in any possible way, they will never be homeomorphic, since a necessary condition in order for two spaces to be homeomorphic, is that they have, as sets, the same cardinality, i.e the same number of elements.

(b) The same argument of (a) shows that a one-point set cannot be homeomorphic to an interval.

(c) More sophisticated arguments are required to show that a closed interval is not homeomorphic to a cross; that is, the topological spaces depicted in the upper part of Figure 0.3 are not homeomorphic to each other. A way of deciding this might be the following: Whatever point we delete from the interval decomposes it in at most two connected portions. However, there is one point in the cross that when deleted, it decomposes the cross into four pieces (components). For that reason, no point in the first space can exist that would correspond to this special point in the second space under a homeomorphism. Therefore, they cannot be homeomorphic.

(d) The surface of a torus is not homeomorphic to that of a sphere. This might be shown if we observe that a circle (a simple closed curve) can be drawn on the surface of the torus that cannot be continuously contracted to a point.

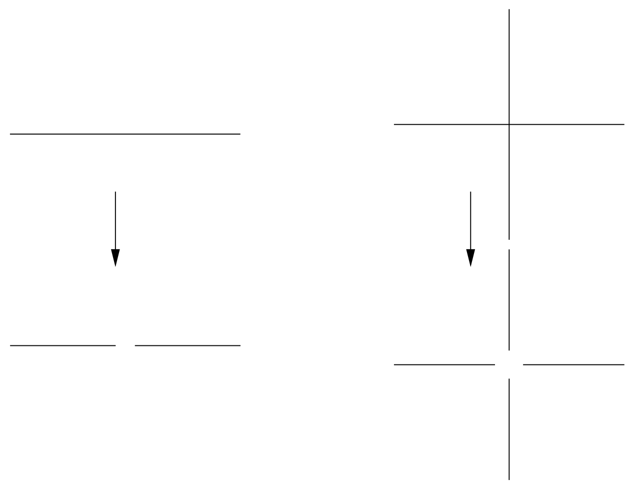


Figure 0.3 An interval and a cross are not homeomorphic

However, it is very clear that any circle drawn in the surface of a sphere can be deformed to a point, as can be appreciated in Figure 0.4. Another way of seeing this might be observing that any circle on the sphere is such that when deleted, the sphere is decomposed in two regions, while the circle on the torus does not have this property. (In other words, the famous Jordan curve theorem holds on the sphere, while it does not hold on the torus.)

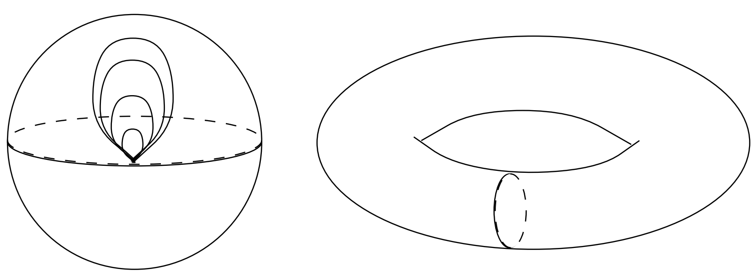


Figure 0.4 Any loop can be contracted on the sphere and a loop that does not contract on the torus

(e) The *Moebius band* that can be obtained from a strip of paper twisting it one half turn and then gluing it along its ends, is not homeomorphic to the *trivial band* obtained from a similar strip by gluing its ends without twisting it (see Figure 0.5).

The argument for showing this can be similar to that used in Example (d), namely there is a circle on the Moebius band, that if removed from the band, does not disconnect it (it can be cut with scissors along the equator

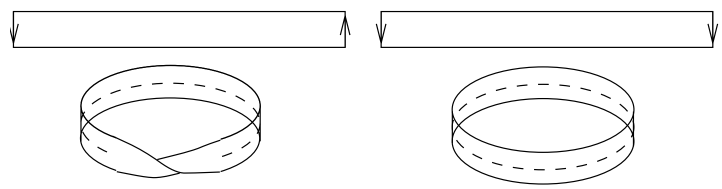


Figure 0.5 The Moebius band and the trivial band

and would not fall apart in two pieces). However, in the trivial band any circle parallel to and different from the edges (or any other circle different from the edges), when removed, decomposes the band in two components (see Figure 0.6).

The first *exercises* for the reader are the following:

(f) Take the Moebius band and cut it along the equator. What space do you obtain? Will it be a Moebius band again? Or maybe will it be the trivial band?

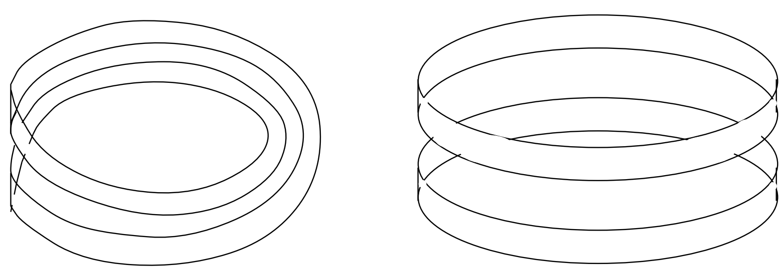


Figure 0.6 The Moebius band is not homeomorphic to the trivial band

(g) Similarly to the given construction of the Moebius band we may take a paper strip and glue its ends, but this time after twisting a full turn. Will this space be homeomorphic to the Moebius band? Or will this space be homeomorphic to the trivial band? What is the relationship between this space and that of (f)? (see Figure 0.6.)

One of the central problems in topology consists, precisely, in studying topological spaces in order to be able to distinguish between them. In all the previous examples every time we have decided that two spaces are not homeomorphic, it has been on the base of certain *invariants* that can be assigned to them. For instance, in (a) this invariant is the cardinality, in (c) it is the number of components obtained after deleting a point, and in (d) and (e) it is the number of components obtained after removing a circle. One of the goals pursued by topology is to assign to each space invariants that are relatively easy to compute and allow to distinguish among them.

When we mentioned the intuitive concept of homeomorphism, we used the intuitive concept of nearness of two points, that is, we talked about the possibility of deciding if a point is close to a given point, or equivalently if it is in a *neighborhood* of the given point. In the first chapters of this book we shall make this concept precise.

A *knot* K is is a simple closed curve in the 3-space, i.e. it is the image $k(\mathbb{S}^1)$ under a "decent" inclusion $k : \mathbb{S}^1 \hookrightarrow \mathbb{R}^3$ of the unit circle in the plane into the 3-dimensional Euclidean space as intuitively shown in Figure 0.7.

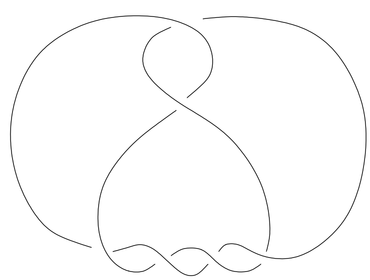


Figure 0.7 A knot in 3-space

Knot theory is an important field of mathematics with striking applications in several branches of science. The central problem of the theory consists in determining when two given knots are *equivalent*; that is, when is it possible to deform inside the space one knot to the other without tearing it apart. A result in the theory proved by Gordon and Luecke [8] states that a knot K is determined by its complement, that is, that two knots K and K' are equivalent if and only if their complements $\mathbb{R}^3 - K$ and $\mathbb{R}^3 - K'$ are homeomorphic. In other words, they transformed the problem of classifying knots into a homeomorphism problem of certain *open sets* in $\mathbb{R}^3$.

Chapter 1 Metric spaces

A very rich source of topological examples are the metric spaces, since, in a very natural way, they have the fundamental topological properties. In this chapter we study briefly the basic concepts of the theory of metric spaces. We start by discussing Euclidean spaces and their subspaces.

1.1 Euclidean spaces

In order to become familiar with the general notation of this book, in this section we present a series of examples of "canonical" topological spaces that play a very important role in topology, as well as in other branches of mathematics. The symbols $\mathbb{R}$ and $\mathbb{C}$ will denote, as usual, the spaces of the real and the complex numbers. If $n \geq 1$, then $\mathbb{R}^n$ will be the *Euclidean space* of dimension n, that is, $\mathbb{R}^n = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}, i = 1, \dots, n\}$ with its usual operations as a vector space: if $x, y \in \mathbb{R}^n$ y $r \in \mathbb{R}$, then $x + y = (x_1 + y_1, \dots, x_n + y_n)$, $x - y = (x_1 - y_1, \dots, x_n - y_n)$ and $rx = (rx_1, \dots, rx_n)$, and the *norm* is given by $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. One defines the *distance* between two points simply by $|y - x|$. $\mathbb{R}^0$ represents the Euclidean space consisting of only one point, the 0, that can be seen as a subspace of the space $\mathbb{R}^n$. There is a canonical identification $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ given by $((x_1, \dots, x_n), (y_1, \dots, y_m)) = (x_1, \dots, x_n, y_1, \dots, y_m)$. Similarly, $\mathbb{R}^n$ can be canonically seen as a subspace of $\mathbb{R}^{n+1}$ identifying it with the subspace $\mathbb{R}^n \times 0$. There is also a canonical identification of $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by setting $(x, y) = x + \mathrm{i}y$, where $\mathrm{i} = \sqrt{-1}$.

1.1.1 Definition. Let $n \geq 0$. Consider the following spaces:

$\mathbb{R}^+ = \{x \in \mathbb{R} \mid 0 \leq x\}$, the *nonnegative halfline.*

$\mathbb{B}^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, the *unit ball* of dimension n, or *unit n-ball.*

$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$, the *unit sphere* of dimension $n - 1$, or *unit $(n-1)$-sphere.*

$\overset{\circ}{\mathbb{B}}{}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$, the *unit cell* or *open unit ball* of dimension n or *unit n-cell.*

$I^n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\, 1 \leq i \leq n\}$, the *unit cube* of dimension n.

$\partial I^n = \{x \in I^n \mid x_i = 0 \text{ or } 1 \text{ for some } i\}$, the *boundary* of I^n in $\mathbb{R}^n$.

$I = I^1 = [0,1] \subset \mathbb{R}$, the *unit interval*.

1.1.2 Exercise. Prove the equality

$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}\,,$$

that is, the equality

$$\mathbb{S}^1 = \{e^{2\pi \mathrm{i} t} \in \mathbb{C} \mid t \in I\}\,.$$

1.2 Metric spaces

In the usual metric on $\mathbb{R}^n$, that we defined above, one considers the concept of *open set*. The concept of a metric and its associated concept of open set can be generalized to any set furnished with a function that behaves analogously to the distance function of $\mathbb{R}^n$. This allows to study a series of properties of those sets with a metric, that are common to those properties of the Euclidean spaces and any of their subspaces. For instance, the concept of a metric is closely related to the concept of convergence of sequences. Frequently there are more general concepts, as that of convergence of a sequence of functions, which require a more general set up than the elementary concept of a Euclidean space. The purpose of this section is to establish an axiomatic system that includes the concepts of convergence, open sets, continuity, etc., that are familiar to us from elementary analysis.

1.2.1 Definition. A *metric space* consists of set X and a function

$$d : X \times X \longrightarrow \mathbb{R}^+,$$

called *metric*, or *distance* that satisfies the following axioms:

(M1) $d(x,y) = 0 \Leftrightarrow x = y$.

(M2) $d(x,y) = d(y,x) \forall x,y \in X$.

(M3) $d(x,y) \leq d(x,z) + d(y,z), \forall x,y,z \in X$.

This last inequality is known as the *triangle inequality* (see figure 1.1).

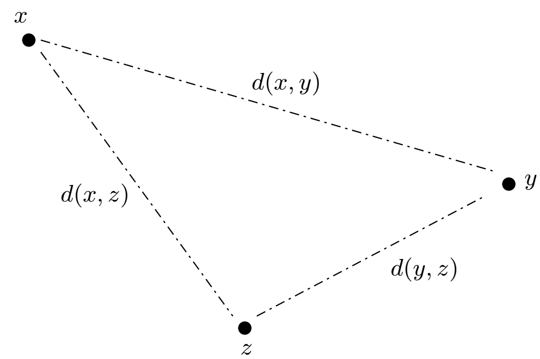


Figure 1.1 The triangle inequality

1.2.2 Examples. The following are metric spaces:

(a) $X = \mathbb{R}$, $d(x, y) = |y - x|$.

(b) $X = \mathbb{R}^n$, $d(x, y) = |y - x|$.

(c) $X \subseteq \mathbb{R}^n$, $d(x, y) = |y - x|$. X is called *metric subspace of* $\mathbb{R}^n$. Examples of metric subspaces of $\mathbb{R}^n$ are the following: $\mathbb{B}^n$, $\overset{\circ}{\mathbb{B}}{}^n$, $\mathbb{S}^{n-1}$, I^n, ∂I^n. From here on, we shall consider these subspaces as metric spaces with this concept.

(d) Let X be any nonempty set and let $d : X \times X \to \mathbb{R}^+$ be given by

$$d(x, y) = \begin{cases} 0, & \text{si } x = y; \\ 1, & \text{si } x \neq y\,. \end{cases}$$

This is clearly a metric which will be called the *discrete metric* on X.

(e) $X = \mathbb{R}^n$, $d(x, y) = \max\{|x_i - y_i| \mid i = 1, \ldots, n\}$.

(f) $X = \{x = (x_i) \mid x_i \in \mathbb{R},\ i \in \mathbb{N},\ \sum_{i=1}^{\infty} x_i^2 < \infty\}$,

$$d(x, y) = \sqrt{\sum_{i=1}^{\infty} (y_i - x_i)^2}$$

This space of *sequences of real numbers* is usually denoted in functional analysis by ℓ^2 and is called the *(real) Hilbert space*.

(g) $X = \{x : I \longrightarrow \mathbb{R} \mid x \text{ is a continuous function}\}$,

$$d(x, y) = \sqrt{\int_0^1 (x(t) - y(t))^2 dt}\,.$$

(h) $X = \{x : I \longrightarrow \mathbb{R} \mid x \text{ is a continuous function}\}$,

$$d(x, y) = \text{máx}\{|x(t) - y(t)| \mid t \in I\}$$

(i) $X = \{x : I \longrightarrow \mathbb{R}^2 \mid x \text{ is a continuous function}, x(0) = x(1)\}$,

$$d(x, y) = \text{máx}\{|x(t) - y(t)| \mid t \in I\}.$$

Observe that in the last example two different functions may have the same image, but however have a nonzero distance between them. For instance, the functions $x, y : I \longrightarrow \mathbb{R}^2$ given by

$$x(t) = (\cos 2\pi t, \sin\, 2\pi t)\,,$$
$$y(t) = (\cos 4\pi t, \sin\, 4\pi t)$$

satisfy $d(x, y) = 2$; however, the image of each of them is $\mathbb{S}^1$.

It is not elementary to prove that the function d in each of the previous examples is indeed a metric. In some cases, certain important (and famous) inequalities are needed. The reader can see [14] or [2], for instance.

Notice that example (d) shows that any set can be given a metric. On the other hand, examples (b) and (e) or (g) and (h) show that the same set may have different nondiscrete metrics. In what follows, $\mathbb{R}^n$ will always denote the metric space of example (b).

1.2.3 REMARK. In the definition of a metric, it is enough to ask that d is a real-valued function and that axioms (M1) and (M3) hold. Axiom (M2) is a consequence of the other two as follows. If we take $z = x$, then we have by (M3) that $d(x, y) \leq d(x, x) + d(y, x) = d(y, x)$, where the last equality holds by (M1). Analogously, $d(y, x) \leq d(x, y)$. Hence $d(x, y) = d(y, x)$ and so (M2) holds. Moreover, if we take $y = x$, then by (M1) and (M3) we have $0 = d(x, x) \leq d(x, z) + d(x, z) = 2d(x, z)$. Thus $d(x, z) \geq 0$ for any x, z and the function d is nonnegative.

1.2.4 **Proposition**. *Let (X, d) be a metric space and consider any subset $Y \subseteq X$. Then the restriction $d' = d|_{Y \times Y} : Y \times Y \longrightarrow \mathbb{R}^+$ is a metric on Y.* □

The metric space Y with the restricted metric will be called a *metric subspace* of X.

1.2.5 EXERCISE. In each of the following cases, prove that the given function $d : X \times X \in \mathbb{R}^+$ is a metric.

(a) In $X = \mathbb{R}^n$ take

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

(b) In $X = \{x : I \longrightarrow \mathbb{R}^n \mid x \text{ is a continuous function}\}$ take

$$d(x,y) = \int_0^1 |x(t) - y(t)|dt.$$

(c) In $X = \{x = (x_i) \mid x_i \in \mathbb{R},\ i \in \mathbb{N},\ x_i \to 0\}$ take

$$d(x,y) = \sup\{|x_i - y_i| \mid i \in \mathbb{N}\}.$$

(d) In $X = \{x = (x_i) \mid x_i \in \mathbb{R},\ i \in \mathbb{N},\ \sum_{i=1}^{\infty} |x_i|^p < \infty\}$, for some $p \geq 1$ take

$$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{\frac{1}{p}}.$$

(*Hint:* Use Minkowski's inequality [2].)

(e) Take X as in (d) and assume $0 < p < 1$. Let d be given by the same formula of (d).

(f) If $d_i : X_i \times X_i \longrightarrow \mathbb{R}$ is a metric on a set X_i, $i = 1, \ldots, n$, take $X = \prod_{i=1}^n X_i$ and take

$$d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i)\,.$$

(g) If $d_i : X_i \times X_i \longrightarrow \mathbb{R}$ is a metric on a set X_i, $i = 1, \ldots, n$, take $X = \prod_{i=1}^n X_i$ and for some $p \geq 1$ take

$$d(x,y) = \left(\sum_{i=1}^{n} d_i(x_i, y_i)^p\right)^{\frac{1}{p}}.$$

The metric spaces of (d) and (e) are usually denoted by ℓ^p. Th metric of (f) is called *product metric*.

1.2.6 DEFINITION. Let X be a real vector space. A *norm* on X is a function that maps each $x \in X$ to a real number $\|x\| \in \mathbb{R}^+$, called the *norm* of x, such that the following conditions hold:

(No1) $\|x\| = 0 \Leftrightarrow x = 0$,

(No2) $\|rx\| = |r| \cdot \|x\|$, $r \in \mathbb{R}$,

(No3) $\|x + y\| \leq \|x\| + \|y\|$.

A vector space X furnished with a norm is called a *normed vector space*.

1.2.7 EXERCISE. Prove the following statements.

(a) In a normed vector space X the function

$$d(x, y) = \|x - y\|$$

defines a metric.

(b) In $\mathbb{R}^n$ each of the following functions defines a norm:

(i) $\|x\| = \text{máx}\{|x_i| \mid i = 1, \ldots, n\}$

(ii) $\|x\| = (\sum_{i=1}^n |x_i|^p)^{1/p}$, $p \geq 1$.

The metric associated to (i) is that of 1.2.2(e) and the metric associated to (ii) is that of 1.2.5(g) if $X_i = \mathbb{R}$ and we take $\mathbb{R}^n = \prod_{i=1}^n \mathbb{R}$.

1.3 NEIGHBORHOODS AND OPEN SETS

As we have already said, in a metric space it is possible to talk about neighborhoods of a point.

1.3.1 DEFINITION. Take $x \in X$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. We define the *open ball* with *center* x and *radius* ε by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Furthermore we say that $U \subset X$ is a *neighborhood of* x if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

The following result is immediate.

1.3.2 **Proposition**. *Let X be a metric space and take a point $x \in X$.*

(a) *If $\varepsilon > 0$, then $B_\varepsilon(x)$ is a neighborhood of x.*

(b) *If U is a neighborhood of x and $U \subseteq V$, then V is a neighborhood of x.* □

1.3.3 DEFINITION. Two metrics d and d' in a set X are said to be *equivalent* if both determine the same neighborhoods of any point $x \in X$.

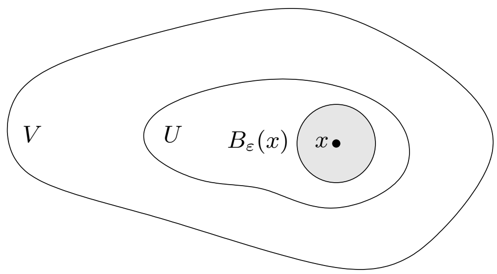


Figure 1.2 A superset of a neighborhood of x is a neighborhood of x

The following statement is easy to prove.

1.3.4 **Proposition**. *Two metrics d and d' in X are equivalent if and only if for each point $x \in X$ the following holds:*

(a) *Given $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that $d'(x,y) < \varepsilon' \Longrightarrow d(x,y) < \varepsilon$, and*

(a) *given $\delta' > 0$, there exists $\delta > 0$ such that $d(x,y) < \delta \Longrightarrow d'(x,y) < \delta'$.*

Proof: First notice that (a) and (b) are equivalent to the next, respectively:

(a') Given $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that $B'_{\varepsilon'}(x) \subseteq B_\varepsilon(x)$, and

(b') given $\delta' > 0$, there exists $\delta > 0$ such that $B_\delta(x) \subseteq B'_{\delta'}(x)$,

where B stands for the balls with respect to the metric d and B' for the balls with respect to d'.

Hence (a) implies that the neighborhoods with respect to d are also neighborhoods with respect to d' and (b) implies that the neighborhoods with respect to d' are also neighborhoods with respect to d. Thus if (a) and (b) hold, then d and d' are equivalent.

Conversely, if d and d' are equivalent and $\varepsilon > 0$, then by 1.3.2 $B_\varepsilon(x)$ is a neighborhood with respect to d', hence (a') holds and thus (a) too. Analogously, $B'_\varepsilon(x)$ is a neighborhood with respect to d and hence (b') holds and thus (b) too.

□

1.3.5 Exercise. Take a metric d in a set X. Show that

(a) the function $d' : X \times X \longrightarrow \mathbb{R}^+$ given by

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

defines a metric on X and this metric is equivalent to d, and

(b) the function $d'' : X \times X \longrightarrow \mathbb{R}^+$ given by

$$d''(x, y) = \min\{d(x, y), 1\}$$

defines another metric on X and this metric is also equivalent to d.

From the previous exercise, we can conclude the following.

1.3.6 **Proposition**. *Every metric space X with metric d has an equivalent metric d' which is bounded.* □

1.3.7 Exercise. Given a metric d on X show that if $k \in \mathbb{R}$, $k > 0$, then the function d' given by $d'(x, y) = kd(X, y)$ defines an equivalent metric on X.

We have also mentioned that the concept of open set in Euclidean spaces can be extended to metric spaces.

1.3.8 Definition. Let X be a metric space. We say that a subset A of X is *open* if A is a neighborhood of x for each $x \in A$. In other words, $A \subseteq X$ is open if and only if for every $x \in A$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.

By definition 1.3.3 we have that the open sets of a metric space depend only on the equivalence class of its metric. That is, equivalent metrics determine the same open sets.

1.3.9 **Proposition**. *The following statements hold:*

(a) *The open ball $B_\varepsilon(x)$ is an open set.*

(b) *A subset $A \subset X$ is open if and only if A is a union of open balls.*

Proof: Statement (a) follows from the triangle inequality and statement (b) follows from the definition of an open set. □

If we consider all open sets in a metric space X, we have the following.

1.3.10 **Theorem**. *Let $\mathcal{A}$ be the set of all open sets in a metric space X. Then the following statements hold:*

(O1) *Let $\mathcal{I}$ be an arbitrary set of indexes. If $\{A_i\}_{i \in \mathcal{I}}$ is a family of elements in $\mathcal{A}$, then $\bigcup_{i \in \mathcal{I}} A_i$ is an element in $\mathcal{A}$.*

(O2) *Let $\mathcal{I}$ be a finite set of indexes. If $\{A_i\}_{i\in\mathcal{I}}$ is a family of elements in $\mathcal{A}$, then $\bigcap_{i\in\mathcal{I}} A_i$ is an element in $\mathcal{A}$.*

In the particular case $\mathcal{I} = \emptyset$, by definition, $\bigcup_{i\in\emptyset} A_i = \emptyset$ and $\bigcap_{i\in\emptyset} A_i = X$. Therefore (O1) *and* (O2) *imply that $\emptyset$ and X belong to $\mathcal{A}$.*

Proof: The sets X and $\emptyset$ are obviously open, that is, they belong to $\mathcal{A}$. The former does because it is the universe where all balls are constructed, and the latter belongs to $\mathcal{A}$ by vacuity. Therefore, we may assume that $\mathcal{I} \neq \emptyset$.

(O1) It follows from 1.3.2.

(O2) Let $\mathcal{I}$ be finite and take $x \in \bigcap_{i\in\mathcal{I}} A_i$. Since each A_i is open, there exists $\varepsilon_i > 0$ such that $B_{\varepsilon_i}(x) \subseteq A_i$, $i \in \mathcal{I}$. Take $\varepsilon = \min\{\varepsilon_i \mid i \in \mathcal{I}\}$. Since $\mathcal{I}$ is finite, $\varepsilon > 0$ and clearly $B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x)$ for all $i \in \mathcal{I}$. Hence we have that $B_\varepsilon(x) \subseteq \bigcap_{i\in\mathcal{I}} A_i$ and thus $\bigcap_{i\in\mathcal{I}} A_i$ is open (see Figure 1.3). □

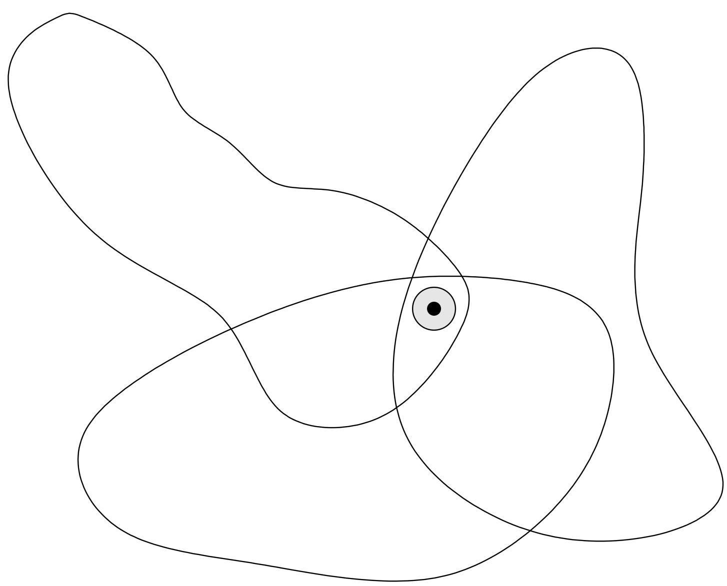


Figure 1.3 The intersection of finitely many open sets is open

1.3.11 Exercise. Take $X = \mathbb{R}^n$ and let

$$d'(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

be the metric of 1.2.5(a).

(a) Show that the neighborhoods of a point in $\mathbb{R}^n$ with the usual metric (Example 1.2.2(b)) are the same as those given by the metric d'.

(b) What can be said about the neighborhoods of a point in $\mathbb{R}^n$ defined using the metric of Example 1.2.2(e)?

(c) Show that in general, if $k = 1, 2, 3, \ldots$, the function

$$d_k(x, y) = \sqrt[k]{\sum_{i=1}^{n} |x_i - y_i|^k}$$

determines a metric and analyze the neighborhoods determined by it. (*Hint:* Use Minkowski's inequality [2].)

1.3.12 Exercise. Take a discrete metric space X as in Example 1.2.2(d). Show that every point in X (seen as a one-point set) is a neighborhood of itself. In other words, with respect to the discrete metric, every point is a neighborhood of itself, hence every point is an open set.

1.3.13 Exercise. Show that all metrics defined in 1.2.5(g) determine the same neighborhoods and prove that these neighborhoods are the same as those determined by the metric of 1.2.5(f).

1.3.14 Exercise. Show that the function $d : I \times I \longrightarrow \mathbb{R}$ given by $d(s,t) = |s-t|^2$ satisfies (M1) and (M2). However, d does not satisfy (M3). A set X together with a function $d : X \times X \longrightarrow \mathbb{R}$ which satisfies (M1) and (M2) is called a *semimetric space*. Thus I with d as defined above is a semimetric space.

1.3.15 Exercise. Let X be a metric space with metric d. A set $C \subseteq X$ is said to be *closed* if its complement $X - C$ is open. We say that a set $B \subseteq X$ is *bounded* if $B \subset B_n(x)$ for some point $x \in X$ and some $n \in \mathbb{N}$. Consider the set $\mathcal{C}(X)$ consisting of all nonempty bounded closed sets in X. For $C \in \mathcal{C}(X)$ and $\varepsilon > 0$ define the *ε-neighborhood* of C to be the set

$$N_\varepsilon(C) = \bigcup \{B_\varepsilon(x) \mid x \in C\}\,.$$

Now define $\delta : \mathcal{C}(X) \times \mathcal{C}(X) \longrightarrow \mathbb{R}$ by

$$\delta(C, D) = \inf\{\varepsilon > 0 \mid C \subset N_\varepsilon(D) \text{ and } D \subset N_\varepsilon(C)\}\,.$$

Prove that δ is a metric that makes $\mathcal{C}(X)$ into a metric space. This metric is called the *Hausdorff metric*.

1.3.16 Exercise. Let X be a metric space with metric d, and take a subset $A \subset X$. Show that A is bounded if and only if there is a positive number R such that $d(x, y) \leq R$ for every pair of elements $x, y \in A$. Define the *diameter* of a bounded set A to be the number $\operatorname{diam} A = \sup\{d(x, y) | x, y \in A\}$. If $\Delta = \operatorname{diam} A$ exhibit a ball (giving its center and radius in terms of A and Δ) that contains A.

1.4 Convergence

Convergence of sequences is an important concept in analysis in Euclidean spaces. This concept can be defined in a similar way in any metric space.

1.4.1 Definition. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in a metric space X. We shall say that (x_n) *converges* to x, written $x_n \to x$, if for every neighborhood V of x there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $x_n \in V$. If a sequence converges to x, we say that it is *convergent.* and we call x the *limit* of the sequence.

1.4.2 Exercise. Show that one obtains the same convergence concept if instead of asking V to be a neighborhood of x, we only require it to be a ball with center in x.

The metric does not play the essential role in this concept of convergence. The concept lies rather on the neighborhoods and not explicitly on the metric. Since there are different metrics that determine the same neighborhoods, we have the same concept of convergence in these metric spaces. In other words, the concept of convergence depends exclusively upon the neighborhoods, or upon the open sets of the space, rather than upon the metric. We shall see below how we can axiomatize the concept of neighborhood without using the concept of metric. Therefore we shall have an associated concept of convergence of sequences.

1.4.3 Examples. In each of the next cases we shall refer to the metric spaces of the Examples 1.2.2 using the corresponding letter.

(a) Convergence in $\mathbb{R}$ with the usual metric is the usual convergence.

(b) Convergence in $\mathbb{R}^n$ with the usual metric is the usual convergence.

(d) Convergent sequences in a discrete metric space are the almost constant sequences, that is, they are the sequences such that $x_n = x$ for all $n > n_0$ (some $n_0 \in \mathbb{N}$).

(e) According to the comments about this metric made above, its associated neighborhoods are the usual neighborhoods in $\mathbb{R}^n$, therefore, convergence in this metric is the usual convergence.

(g) Here we have a concept of convergence of functions in terms of integrals.

Consider a sequence of functions (f_n), where for each n, $f_n : I \longrightarrow \mathbb{R}^n$. The sequence is said to be *pointwise convergent* to a function $f : I \longrightarrow \mathbb{R}^n$, if for each $t \in I$ the sequence $(f_n(t))$ of points in $\mathbb{R}^n$ converges in the usual way in $\mathbb{R}^n$ to the point $f(t)$. Consider the following question: Does there exist a metric on the set $X = \{x : I \longrightarrow \mathbb{R} \mid x \text{ is a continuous function}\}$ that induces pointwise convergence? The answer is no.

We shall give below a more general concept of convergence that will depend rather on the neighborhoods and not on the metric. Dependance on the metric makes the concept much too restrictive. This is one of the several reasons why it is convenient to axiomatize the concept of open set in a metric space, to put aside all problems related to the metric.

1.5 PSEUDOMETRIC SPACES

Before passing to the axiomatization of the structure of the open sets in a metric space, it is convenient to introduce a generalization of the concept of metric space, which shares several of its properties and is frequently used as a source of examples. It has the same structure of its open sets.

1.5.1 DEFINITION. Let X be a set and let $d : X \times X \longrightarrow \mathbb{R}$ be a function that fulfills axiom (M3) of a metric and, instead of axiom (M1), it satisfies the axiom

(SM1) $d(x,x) = 0 \ \forall x \in X$.

Such a function is called a *pseudometric* in X and X together with d is called a *pseudometric space*.

Notice that in a pseudometric space the "pseudodistance" between two points may be zero even though the two points are different.

1.5.2 REMARK. As in Remark 1.2.3, one may show that if axioms (SM1) and (M3) hold, then $d(x,y) \geq 0$ for all $x, y \in X$, and axiom (M2) holds.

1.5.3 EXERCISE. Show that for every $X \neq \emptyset$, the function $d : X \times X \longrightarrow \mathbb{R}^+$ given by $d(x, y) = 0$ for any $x, y \in X$ is a pseudometric. This is the so-called *indiscrete pseudometric.*

In the same way as in the case of metric spaces, we can define open sets in pseudometric spaces. Let X be a pseudometric space with a pseudometric d and take a subset $A \subset X$. We say that A is *open* if for each point $x \in A$ there exists $\varepsilon > 0$, such that the *open pseudoball* $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ with *center* x and *radius* ε lies in A.

1.5.4 EXERCISE. Show that the collection $\mathcal{A}$ of open sets in a pseudometric space X satisfies conditions (O1) and (O2) in 1.3.10.

1.5.5 EXERCISE.

(a) Let X be a pseudometric space with pseudometric d and let $\sim$ be the relation given by $x \sim y \Leftrightarrow d(x, y) = 0$. Show that $\sim$ is an equivalence relation in X. In the set of equivalence classes $\widetilde{X} = X/\sim$ consider the function

$$\widetilde{d} : \widetilde{X} \times \widetilde{X} \to \mathbb{R}^+,$$

given by $\widetilde{d}([x], [y]) = d(x, y)$, where $[x], [y]$ denote the corresponding equivalence classes. Show that $\widetilde{d}$ is a well-defined metric on $\widetilde{X}$. The metric space $\widetilde{X}$ with the metric $\widetilde{d}$ is called *metric identification* of X with respect to the pseudometric d.

(b) How are the open sets of $\widetilde{X}$ related to those of X? Give a characterization of them.

1.5.6 REMARK. Pseudometrics arise naturally in functional analysis. For instance, consider the space $\mathcal{F}(X)$ of real-valued functions $f : X \longrightarrow \mathbb{R}$ together with a special point x_0 in the set X. This point then induces a pseudometric on the space of functions, given by

$$d(f, g) = |f(x_0) - g(x_0)|$$

for $f, g \in \mathcal{F}(X)$.

1.5.7 EXERCISE.

(a) Define an equivalence relation in the square $I \times I$ in such a way, that it identifies points of the form $(0, t)$ with those of the form $(1, t)$. This relation corresponds to gluing the left edge of the square with the right edge to obtain a cylinder. Construct a pseudometric on $I \times I$, whose metric identification provides the cylinder with a metric. (*Hint:* Let d be the usual metric on the plane $\mathbb{R}^2$ and for $x, Y \in I \times I$, define $d'(x,y) = \min\{d(x,y), d(x+(1,0),y)\}$. Check that d' is a pseudometric with the desired property. The induced metric measures the distance between two points of the cylinder in a "geodesic" way, that is, "walking" over the surface of the cylinder.)

(b) Extend the equivalence relation of (a) to include now the equivalence of a point of the form $(s,0)$ with the point $(s,1)$. This new relation corresponds to gluing the left edge of the square with the right edge, as well as the bottom edge with the top edge, to obtain a torus. Construct a pseudometric on $I \times I$, whose metric identification provides the torus with the "geodesic" metric. (*Hint:* Let d be the usual metric on the plane $\mathbb{R}^2$ and for $x, y \in I \times I$, define $d'(x,y) = \min\{d(x,y), d(x+(1,0),y), d(x+(0,1),y), d(x+(1,1),y)\}$. Check that d' is a pseudometric with the desired property. This is the "geodesic" metric on the torus.)

(c) Define an equivalence relation in the square $I \times I$ in such a way, that it identifies points of the form $(0,t)$ with those of the form $(1, 1-t)$. This relation corresponds to gluing the left edge of the square with the right edge, but in opposite directions, to obtain a Moebius band. Construct a pseudometric on $I \times I$, whose metric identification provides the Moebius band with a metric. (*Hint:* Let d be the usual metric on the plane $\mathbb{R}^2$ and for $x = (s_1, s_2)$, $y = (t_1, t_2) \in I \times I$, define $d'(x,y) = \min\{d(x,y), d(x',y), d(x,y')\}$, where $x' = (s_1 + 1, 1 - s_2)$ and $y' = (t_1 + 1, 1 - t_2)$.)

1.5.8 Exercise.

(a) Prove that if $X = \{x : I \longrightarrow \mathbb{R} \mid x \text{ is an integrable function}\}$, then

$$d(x,y) = \sqrt{\int_0^1 (x(t) - y(t))^2 dt}$$

defines a pseudometric.

(b) Describe the metric identification of the pseudometric space of (a).

1.5.9 Exercise. Notice that Definitions 1.3.1 and 1.3.8 make sense for pseudometric spaces and prove Proposition 1.3.10 in this more general case.

1.5.10 Exercise. Prove that for the indiscrete pseudometric (see 1.5.3) the only open sets are X and $\emptyset$. What is the metric identification in this case?

1.5.11 Exercise. Let X be a set, Y a metric space with a metric d' and $f : X \longrightarrow Y$ an arbitrary function. Show that the function $d : X \times X \longrightarrow \mathbb{R}$ given by

$$d(x, y) = d'(f(x), f(y))$$

is a pseudometric. When is d a metric?

Show that the statement is equally valid if d' is only a pseudometric.

1.5.12 Exercise. Do the following functions define a metric on $\mathbb{R}^2$? Do they define a pseudometric?

(a) $d((x_1, x_2), (y_1, y_2)) = |y_1 - x_1|$.

(b) $d((x_1, x_2), (y_1, y_2)) = |x_1 x_2 - y_1 y_2|$.

(c) $d((x_1, x_2), (y_1, y_2)) = |x_1 + y_1 - x_2 - y_2|$.

Chapter 2 Topological spaces

In this chapter we shall give the basic definitions of topological space, open and closed sets, neighborhoods, and all definitions related with the description of a topological space. All these concepts are axiomatically defined taking as a model the corresponding concepts that one has in metric or pseudometric spaces. This way, metric and pseudometric spaces will be examples of topological spaces.

2.1 Basic definitions: open sets and neighborhoods

In Proposition 1.3.10 of the previous chapter, two properties of the family of open sets of a metric space were proved, namely (O1) and (O2). Further, in Exercise 1.5.4 we asked to prove the same properties for the open sets of a pseudometric space. They suggest what the basic definition of a topological space should be. This will be the starting point of the rest of this book.

2.1.1 Definition. Let X be a set. A *topology* in X is a family $\mathcal{A}$ of subsets of X, called the *open sets*, which satisfies the following two axioms:

(O1) If $\{A_i\}_{i\in\mathcal{I}} \subseteq \mathcal{A}$, where $\mathcal{I}$ is an arbitrary set, then $\bigcup_{i\in\mathcal{I}} A_i \in \mathcal{A}$.

(O2) If $\{A_i\}_{i\in\mathcal{I}} \subseteq \mathcal{A}$, where $\mathcal{I}$ is a finite set, then $\bigcap_{i\in\mathcal{I}} A_i \in \mathcal{A}$.

In particular, since X is the intersection of an empty family, X lies in $\mathcal{A}$. Moreover, since $\emptyset$ is the union of an empty family, $\emptyset$ also lies in $\mathcal{A}$. The pair $(X, \mathcal{A})$ is called a *topological space*, which will be denoted simply by X if there is no danger of confusion with respect to the topological structure given by its open sets. The set X will be called the *underlying set* of the topological space.

2.1.2 Examples. The following are topologies.

(a) The family $\mathcal{A}$ consisting of all subsets of a set X. This is called the *discrete topology* in X. The corresponding topological space is called *discrete space*. Therefore, in a discrete space X, every subset of X is open.

(b) The family $\mathcal{A}$ consisting only of $\emptyset$ and X. This is called the *indiscrete topology* in X (this is also called the *trivial topology*. The corresponding topological space is called *indiscrete space*. Therefore, in an indiscrete space X, the only open sets are $\emptyset$ and X.

(c) The family $\mathcal{A}$ of all open sets of a metric space X. This is called a *metrizable topology*. The corresponding topological space is usually called *metrizable space*. In particular, if X has the discrete metric, then $\mathcal{A}$ is the discrete topology, and X is a discrete space. Therefore, *discrete spaces are metrizable*.

(d) The family $\mathcal{A}$ of all open sets of a pseudometric space X. This is called a *pseudometrizable topology*. The corresponding topological space is called *pseudometrizable space*. In particular, if X has the indiscrete pseudometric, then $\mathcal{A}$ is the indiscrete topology, and X is an indiscrete space. Therefore, *indiscrete spaces are pseudometrizable*. Notice that indiscrete spaces are not metrizable, unless they have at most one point.

(e) The family $\mathcal{A}$ consisting of all subsets of a set X whose complement is finite and of $\emptyset$. If X es finite, then this topology is clearly the discrete topology. If X is infinite, then we call it the *cofinite topology*.

(f) The family $\mathcal{A}$ consisting of all subsets of a set X whose complement is at most countable and of $\emptyset$. If X is countable, then this topology is clearly the discrete topology. If X es uncountable, then we call it the *cocountable topology*.

2.1.3 Exercise. Let X be a topological space and take a fixed subset $S \subset X$. Prove that the set $\{A \cup (B \cap S) \mid A, B \text{ are open sets in } X\}$ is another topology on the underlying set of X.

By the discussion of the previous chapter, we know that different metrics on the same set X may determine the same open sets and therewith, also the same topology, as it was the case in Exercise 1.3.11, for instance. In two equivalent metrics, convergence of sequences is the same, i.e. a sequence is convergent in one metric if and only if it is convergent in the other. This suggests that convergence is a topological concept, rather than a metric concept.

2.1.4 Definition. Given a sequence (x_n) in a topological space, we say that it *converges* to a point x, if for every open set A containing x, the sequence lies inside A from certain element on. In symbols, $x_n \to x$ if for each open set A such that $x \in A$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$. We say that x is a *limit* of the sequence.

2.1.5 Exercise. Analyze and describe the convergence of sequences in all the topologies of the examples 2.1.2.

As it is the case with metric spaces, in topological spaces we also can define the concept of neighborhood.

2.1.6 Definition. Let X be a topological space and take $x \in X$. We define a *neighborhood* of x to be a set $U \subseteq X$ such that there exists an open set A in X that satisfies $x \in A \subseteq U$.

This definition is clearly consistent with the one given for metric spaces 1.3.1, since an open ball in a metric space is an open set. Hence a neighborhood of a point in a metric space is a neighborhood of the same point in the topological space determined by the metric, and conversely. The definition of an open set in a metric space 1.3.8 becomes, for topological spaces, the following result.

2.1.7 **Theorem**. *Let X be a topological space. A subset $A \subseteq X$ is open if and only if A is a neighborhood of every $x \in A$.*

Proof: If A is an open set, then it is clearly a neighborhood of all its points.

Conversely, assume that A is a neighborhood of each of its points, and take $x \in A$. Since A is a neighborhood of x, there exists an open set A_x such that $x \in A_x \subseteq A$. Therefore, $A = \bigcup_{x \in A} A_x$, and by axiom (O1), A is open. □

From here on we shall denote by $\mathcal{N}_x^X$ the set of all neighborhoods of a point x in a topological space X. When there is no danger of confusion, we denote this set simply by $\mathcal{N}_x$. The family $\mathcal{N} = \{\mathcal{N}_x \mid x \in X\}$ will be called the *neighborhood system* for the topology of X.

2.1.8 Note. One may restate the definition of convergence given in Exercise 2.1.5 by saying that a sequence (x_n) in a topological space X converges to x, in symbols $x_n \to x$, if for every $V \in \mathcal{N}_x$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq n_0$.

2.1.9 **Proposition**. *The neighborhood system $\mathcal{N}$ for the topology of X has the following properties.*

(N1) $V \in \mathcal{N}_x$, $V \subseteq U \Rightarrow U \in \mathcal{N}_x$.

(N2) $V_i \in \mathcal{N}_x$, $i \in \mathcal{I}$, $\mathcal{I}$ *finite*, $\Rightarrow \bigcap_{i \in \mathcal{I}} V_i \in \mathcal{N}_x$.

(N3) $V \in \mathcal{N}_x \Rightarrow x \in V$.

(N4) $U \in \mathcal{N}_x \Rightarrow \exists\, V \in \mathcal{N}_x$ *such that* $U \in \mathcal{N}_y\, \forall\, y \in V$.

Proof:

(N1) It is clear.

(N2) It follows from (O2).

(N3) It is obvious by definition.

(N4) Let V be an open set in X such that $x \in V \subseteq U$. Therefore V is a neighborhood of x and for each $y \in V$, $y \in V \subseteq U$. Thus U is a neighborhood of y. □

2.1.10 Definition. Take a set X and let $\mathcal{N}_x$, $x \in X$, be a family that fulfills conditions (N1)–(N4) of the previous proposition. Such a family $\mathcal{N} = \{\mathcal{N}_x \mid x \in X\}$ is called a *neighborhood system* on X.

A neighborhood system on a set X determines a topology on X as follows.

2.1.11 **Theorem**. *Let X be a set and $\mathcal{N}$ a neighborhood system in X. Then there exists a unique topology on X that has $\mathcal{N}$ as its neighborhood system.*

Proof: First we prove that if such topology exists, then it is unique. Indeed, Theorem 2.1.7 states that a set $A \subseteq X$ is open if and only if A is a neighborhood of each of its points. In other words, A is open if and only if $A \in \mathcal{N}_x$ for all $x \in A$. This fact characterizes the open sets uniquely. Hence the topology must be unique.

This allows us now to define the topology. Indeed, define

$$\mathcal{A} = \{A \subseteq X \mid A \in \mathcal{N}_x\ \forall x \in A\}.$$

We now have to prove that $\mathcal{A}$ is in fact a topology. We must check that the axioms hold.

First notice that $\emptyset \in \mathcal{A}$ by vacuity, and $X \in \mathcal{A}$ by Axiom (N1).

(O1) Let $\{A_i\}_{i \in \mathcal{I}} \subseteq \mathcal{A}$ be a nonempty family of sets in $\mathcal{A}$. If $x \in \bigcup_{i \in \mathcal{I}} A_i$, then $x \in A_i$ for some $i \in \mathcal{I}$. Since $A_i \in \mathcal{A}$, $A_i \in \mathcal{N}_x$. By (N1), $\bigcup_{i \in \mathcal{I}} A_i \in \mathcal{N}_x$ and therefore $\bigcup_{i \in \mathcal{I}} A_i \in \mathcal{A}$.

(O2) Let $\{A_i\}_{i\in\mathcal{I}} \subseteq \mathcal{A}$ be a finite nonempty family of sets in $\mathcal{A}$. If $x \in \bigcap_{i\in\mathcal{I}} A_i$, then $x \in A_i$ for all $i \in \mathcal{I}$. Since $A_i \in \mathcal{A}$, $A_i \in \mathcal{N}_x$ for all $i \in \mathcal{I}$. By (N2), $\bigcap_{i\in\mathcal{I}} A_i \in \mathcal{N}_x$ and therefore $\bigcap_{i\in\mathcal{I}} A_i \in \mathcal{A}$.

We have shown that $\mathcal{A}$ is a topology on X. We now have to study the neighborhoods in this topology. We shall prove that they are precisely the elements of the given neighborhood system. Namely, let V be a neighborhood of x with respect to the topology $\mathcal{A}$, that is, there exists a set $A \in \mathcal{A}$ such that $x \in A \subseteq V$. By definition of $\mathcal{A}$, one has $A \in \mathcal{N}_x$, and by (N1), $V \in \mathcal{N}_x$. Conversely, assume that $U \in \mathcal{N}_x$. Define $A = \{y \in X \mid U \in \mathcal{N}_y\}$. In particular, $x \in A$ and, by (N3), $y \in U$ for all $y \in A$. Thus $x \in A \subseteq U$. Hence it is enough to check that $A \in \mathcal{A}$.

Take $y \in A$. Since $U \in \mathcal{N}_x$, then by definition of A and by (N4) we know that there exists $V \in \mathcal{N}_y$ such that $U \in \mathcal{N}_z$ for all $z \in V$. Hence $V \subseteq A$ and by (N1), A is a neighborhood of y. Therefore $A \in \mathcal{A}$ as desired. □

The construction given in the proof of 2.1.11 of the open sets A leads to the next.

2.1.12 Definition. Let X be a topological space and take a subset $A \subseteq X$. Define the *interior* of A as the set

$$A^\circ = \{x \in A \mid A \in \mathcal{N}_x\}\,.$$

A point $x \in A^\circ$ is called an *interior point* of A.

2.1.13 **Theorem**. *Let X be a topological space and A be a subset of X. Then*

$$A^\circ = \bigcup\{B \subseteq A \mid B \textit{ is open in } X\}.$$

Therefore A° is an open set. Indeed, A° is the largest open set of X contained in A.

Proof: Take $x \in A^\circ$. Then A is a neighborhood of x. Hence there is an open set $B \subset X$ such that $x \in B \subseteq A$. Thus, clearly, $A^\circ \subseteq \bigcup\{B \subseteq A \mid B \text{ is open in } X\}$.

Conversely, take $x \in \bigcup\{B \subseteq A \mid B \text{ is open in } X\}$. Then $x \in B \subseteq A$ for some open set $B \subset X$. Therefore $A \in \mathcal{N}_x$, that is, $x \in A^\circ$. Hence, $\bigcup\{B \subseteq A \mid B \text{ is open in } X\} \subseteq A^\circ$. □

2.1.14 **Corollary**. *A is open in X if and only if $A = A^\circ$.* □

This corollary is an equivalent statement of Theorem 2.1.7, which was proved above.

2.1.15 DEFINITION. Let X be a topological space and A be a subset of X. The assignment $A \mapsto A^\circ$ is called *interior operator* of the topology of X.

It is easy to prove the following.

2.1.16 **Theorem**. *Let X be a topological space. The interior operator has the following properties:*

(I1) $X^\circ = X$.

(I2) $A^\circ \subseteq A$.

(I3) $(A^\circ)^\circ = A^\circ$.

(I4) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(I5) $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$.

(I6) $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$. □

2.1.17 EXERCISE.

(a) Prove the previous theorem.

(b) Show with an example that, in general, the equality in (I6) does not hold.

(c) Prove that (I5) and (I6) are consequences of (I4).

2.1.18 DEFINITION. Let X be a set. An operator in the subsets of X, $A \mapsto A^\circ$, that satisfies (I1)–(I4), is called *interior operator* on the set X.

In the same spirit of Theorem 2.1.11, that characterizes the topology of a topological space in terms of its neighborhoods, that is, in terms of the axioms (N1)–(N4), we have the following.

2.1.19 **Theorem**. *Let X be a set and let $A \mapsto A^\circ$ be an interior operator on X. Then there exists a unique topology $\mathcal{A}$ on X, whose interior operator is the given one.*

Proof: Define $\mathcal{A} = \{A \subseteq X \mid A = A^\circ\}$. Then $\mathcal{A}$ is the desired topology on X. □

2.1.20 EXERCISE. Prove that indeed $\mathcal{A}$, as defined in the previous proof, is a topology. Furthermore, prove that it is the only topology on X that has $A \mapsto A^\circ$ as its interior operator.

2.1.21 Exercise. Let X be a set and and take a fixed subset $A_0 \subseteq X$. Define an operator $A \mapsto A^\circ$ by

$$A^\circ = \begin{cases} X & \text{if } A = X\,, \\ A - A_0 & \text{if } A \neq X\,, \end{cases}$$

where $A - A_0 = \{x \in A \mid x \notin A_0\}$. Prove that this is an interior operator on X. Analyze this operator in the cases $A_0 = X$ or $A_0 = \emptyset$ and explain what topology is described in each case.

2.1.22 Exercise. Describe the interior operator for the cofinite, resp. the co-countable, topology on an infinite, resp. uncountable, set X.

2.2 Closed sets

The subsets of a topological space X that we shall call "closed" determine the topology, just as the open sets do. However, their properties are quite different to those of the open sets. Therefore, it is convenient to study them separately.

2.2.1 Definition. A point x in a topological space X is called *limit point* of a set $A \subseteq X$ if every neighborhood V of x in X meets A, i.e. is such that $V \cap X \neq \emptyset$. Let

$$\overline{A} = \{x \in X \mid x \text{ is a limit point of } A\}\,.$$

The set $\overline{A}$ is called *closure* a A.

It is easy to prove have the following.

2.2.2 **Proposition**. *The following two conditions hold and are equivalent:*

(a) $\overline{A} = X - (X - A)^\circ$

(b) $X - \overline{A} = (X - A)^\circ$. □

2.2.3 Remark. From the previous result we have that, if $X - A$ is open, then

$$X - A = (X - A)^\circ = X - \overline{A}$$

and hence $\overline{A} = A$.

2.2.4 Definition. A subset A of a topological space X is said to be *closed* if $X - A$ is open.

Thus *a subset A of a topological space X is closed if and only if $A = \overline{A}$.* In particular, we have that X and $\emptyset$ simultaneously open and closed.

From the axioms (O1) and (O2) for the open sets of a topological space one has the following.

2.2.5 **Theorem**. *Let X be a topological space. Then the family $\mathcal{C}$ of the closed sets in X satisfies the following axioms:*

(C1) $B_i \in \mathcal{C}$, $i \in \mathcal{J}$, $\Rightarrow \bigcap_{i\in\mathcal{J}} B_i \in \mathcal{C}$.

(C2) $B_i \in \mathcal{C}$, $i \in \mathcal{J}$, $\mathcal{J}$ *finite,* $\Rightarrow \bigcup_{i\in\mathcal{J}} B_i \in \mathcal{C}$. □

2.2.6 Exercise. Prove the previous theorem with all details.

2.2.7 Examples.

1. In a discrete space all subsets are open and closed, while in an indiscrete space, the only closed subsets are X and $\emptyset$.

2. In the real line $\mathbb{R}$ with the usual topology one has, for instance, that the singular sets $\{t\} \subset \mathbb{R}$, as well as $[a,b]$, $(-\infty, b]$, and $[a, +\infty)$ are closed subsets. On the other hand, an open interval (a,b), $a < b$, is open and its closure is the closed interval $[a,b]$. However,
$$(a,b) = \bigcup_{a<t<b} \{t\}$$
is not closed. This shows that we may not omit the finiteness assumption in (C2).

3. Take the unit interval $I = [0,1]$ and define closed subsets
$$A_1 \supset A_2 \supset \cdots A_{n-1} \supset A_n \supset \cdots$$
in I as follows. A_1 is obtained removing from I the *open middle third*, namely the open interval $(\frac{1}{3}, \frac{2}{3})$ leaving the union of the two closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. A_2 is obtained removing from A_1 the open middle third of each of these remaining segments leaving the union of the four closed intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, and $[\frac{8}{9}, 1]$. This process is continued ad infinitum, where A_n is obtained from A_{n-1} removing the open middle third from each of its 2^{n-1} closed intervals. In other words,
$$A_n = A_{n-1} - \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) .$$

(Notice that some of the intervals on the right-hand side are superfluous, since only 2^{n-1} of the 3^{n-1} intervals are relevant.) The *Cantor set* is the subspace $C = \bigcap_n A_n \subset I$. Since each A_n is closed in I, by (C2) their intersection is closed.

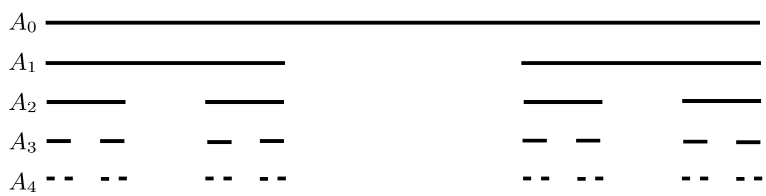


Figure 2.1 Cantor set: First four stages

2.2.8 **Theorem**. *Let X be a set and let $\mathcal{C}$ be a family of subsets of X that satisfies axioms* (C1) *and* (C2). *Then there exists a unique topology $\mathcal{A}$ on X such that $\mathcal{C}$ is precisely the family of closed sets in X.*

Proof: Take $\mathcal{A} = \{A \subseteq X \mid X - A \in \mathcal{C}\}$. Then $\mathcal{A}$ the desired topology on X. □

As a consequence of 2.2.8, we have that the topology of a space X can be characterized by giving its closed sets (instead of its open sets).

2.2.9 Exercise. Prove that indeed $\mathcal{A}$, as defined in the previous proof, is a topology. Furthermore, prove that it is the only topology on X that has $\mathcal{C}$ as its family of closed sets.

2.2.10 Exercise. Give examples of families of closed sets in a topological space, whose union is not a closed set.

2.2.11 Exercise. Prove the following two theorems.

As one can see in the equivalent properties 2.2.2 (a) and (b), the closure concept is, in some sense, *dual* to that of interior of a set; thus, dually to 2.1.13, we have the following result.

2.2.12 **Theorem**. *Let X be a topological space and A a subset of X. Then*

$$\overline{A} = \bigcap\{B \subset X \mid A \subset B \in \mathcal{C}\}\,.$$

Therefore, $\overline{A}$ is a closed set. Indeed, $\overline{A}$ is the smallest closed set that contains A.

□

2.2.13 DEFINITION. Let X be a topological space. The assignment $A \mapsto \overline{A}$, for any $A \subseteq X$, is called the *closure operator* of the topology of X.

Dually to 2.1.16, we have the following.

2.2.14 **Theorem**. *Let X be a topological space. The closure operator in X has the following properties:*

(Cl1) $\overline{\emptyset} = \emptyset$

(Cl2) $\overline{A} \supset A$

(Cl3) $\overline{\overline{A}} = \overline{A}$

(Cl4) $\overline{A \cup B} = \overline{A} \cup \overline{B}$. □

The properties (Cl1)–(Cl-4) of a closure operator are dual to the corresponding properties (I1)–(I4) of an interior operator. Analogously to Exercise 2.1.17 (c) and (b) one can solve the next.

2.2.15 EXERCISE.

(a) Prove that as a consequence of (Cl4) one obtains properties

(Cl5) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$.

(Cl6) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(b) Show with an example that, in general, the equality in (Cl6) does not hold.

2.2.16 EXERCISE. Let $\mathcal{A}$ be the family of all intervals $I_a = (a, \infty)$ in $\mathbb{R}$, where $I_\infty = \emptyset$, and $I_{-\infty} = \mathbb{R}$. Show that $\mathcal{A}$ is a topology on $\mathbb{R}$. In this topology, what is the closure of a set $A \subseteq \mathbb{R}$?

2.2.17 DEFINITION. Let X be a set. An operator in the subsets of X, $A \mapsto \overline{A}$, that satisfies (Cl1)–(Cl4), is called *closure operator* on X. Properties (Cl1)–(Cl4) are known as *Kuratowski axioms*.

Dually to 2.1.19 we have the following.

2.2.18 **Theorem**. *Let X be a set and let $A \mapsto \overline{A}$ be a closure operator on X. Then there exists a unique topology on X, whose closure operator coincides with the given one.*

Proof: The family $\mathcal{C} = \{A \subseteq X \mid A = \overline{A}\}$ satisfies (C1) and (C2). Therefore, by 2.2.8, $\mathcal{C}$ determines a unique topology with it as its family of closed sets. This is the desired topology. □

2.2.19 Exercise. Verify that, indeed, the topology defined in the previous proof is the desired one.

2.2.20 Exercise. Let X be a set and take a fixed subset $A_0 \subset X$, if $\mathcal{P}(X)$ denotes the power of X, define an operator on $\mathcal{P}(X)$, $A \mapsto \widetilde{A}$, given by

$$\widetilde{A} = \begin{cases} A \cup A_0 & \text{if } A \neq \emptyset \\ \emptyset & \text{if } A = \emptyset\,. \end{cases}$$

Prove that this is a closure operator on X. Give the corresponding interior operator and describe the open sets of the topology on X generated by them. Analyze this operator in the cases $A_0 = X$ or $A_0 = \emptyset$ and explain what topology is described in each case. (cf. 2.1.21).

2.2.21 Exercise. Describe the closure operator for the cofinite, resp. the co-countable, topology on an infinite, resp. uncountable, set X.

2.2.22 Exercise. Let X be a topological space and take a subset $A \subset X$. We say that A is *regular open* if $A = (\overline{A})^\circ$, and we say that A es *regular closed* if $A = \overline{(A^\circ)}$. Prove the following statements:

(a) The complement of a regular open set is regular closed, and conversely.

(b) There are open sets that are not regular open (and hence there are closed sets that are not regular closed). Give examples.

(c) For any $A \subset X$, the set $A = (\overline{A})^\circ$ is regular open and the set $A = \overline{(A^\circ)}$ is regular closed.

(d) The intersection of two regular open sets is a regular open set. But the union of two regular open sets is not necessarily a regular open set. Give an example for this assertion.

(e) The union of two regular closed sets is a regular closed set. But the intersection of two regular closed sets is not necessarily a regular closed set. Give an example for this assertion.

2.2.23 Exercise. Take $\mathbb{R}^n$ and declare a set $A \subset \mathbb{R}^n$ as *Zariski-closed* if there exists a polynomial function $p : \mathbb{R}^n \longrightarrow \mathbb{R}$, that is, a function such that $p(x)$ is a polynomial on the components of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $A = p^{-1}(C)$, wher $C \subset \mathbb{R}$ es finite or $\mathbb{R}$.

(i) Prove that the family of arbitrary intersections of Zariski-closed set in $\mathbb{R}^n$ satisfies axioms (C1) and (C2). Hence it determines a topology on $\mathbb{R}^n$. This is the so-called *Zariski topology* on $\mathbb{R}^n$.

(ii) Prove that the Zariski topology on $\mathbb{R}$ is the cofinite topology (see 2.1.2 (e)).

(iii) Consider the cofinite topology on $\mathbb{R}$ and the Zariski topology on $\mathbb{R}^n$. Prove that a polynomial function $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ is is continuous. In fact, prove that the Zariski topology is the coarsest topology on $\mathbb{R}^n$, that is, the topology containing the least number of open sets (see 3.1.1), such that all polynomial functions are continuous, provided that $\mathbb{R}$ has the cofinite topology.

(iv) A subset $\mathcal{V} \subset \mathbb{R}^n$ is called an *algebraic variety* if there exists a polynomial function $p : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that $\mathcal{V} = p^{-1}(0)$. One defines a topology on $\mathcal{V}$ by declaring that the closed sets of $\mathcal{V}$ are precisely the intersections $\mathcal{V} \cap A$, where $A \subset \mathbb{R}^n$ is Zariski-closed. Prove that these closed sets determine, indeed, a topology on $\mathcal{V}$. This is the so-called *Zariski topology* on the algebraic variety $\mathcal{V}$.

(v) If $\mathcal{V}$ is an algebraic variety, we say that a subset of $\mathcal{V}$ is *Zariski-closed* in $\mathcal{V}$ if it is the intersection of a Zariski-closed set in $\mathbb{R}^n$ with $\mathcal{V}$. Prove that $B \subset \mathcal{V}$ is a Zariski-closed set if and only if there exists a polynomial function $q : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $B = q^{-1}(0) \cap \mathcal{V}$.

2.3 Other basic concepts

There are several concepts related to the points of a topological space, which not necessarily characterize it. Notwithstanding, they play an important role as well in the theory as in the applications.

2.3.1 Definition. Let X be a topological space and take a subset $A \subset X$. A point $x \in \overline{A} \cap \overline{X - A}$ is called *boundary point* (or *frontier points*) of A (and of $X - A$) in X. The set $\overline{A} \cap \overline{X - A}$ is called the *boundary* (or the *frontier*) of A (and of $X - A$) and is denoted by ∂A.

2.3.2 Remark.

(i) In general, the boundary of a set A is not contained in A.

(ii) The boundary of a set A is closed.

2.3.3 Exercise. Prove that a set A is closed if and only if A contains its boundary.

Given a set A in a topological space X, there are three types of points with respect to A, namely:

(i) those of the interior of A, A°,

(ii) those of the boundary of A, ∂A,

(iii) and those of the *exterior* of A, $(X - A)^\circ = X - \overline{A}$.

2.3.4 Exercise. Prove that $X = A^\circ \cup \partial A \cup (X - A)^\circ$, and that these three sets are pairwise disjoint.

2.3.5 Exercise. Show that ∂A always contains $\partial(A^\circ)$. How does $\partial(A \cup B)$ relate to ∂A and ∂B?

2.3.6 Definition. Let X be a topological space and let A be a subset of X. A point $x \in A$ is said to be *isolated* if there is a neighborhood U of x in X such that $U \cap A = \{x\}$. Intuitively, an isolated point x of a set A in a topological space X looks like the one shown in Figure 2.2.

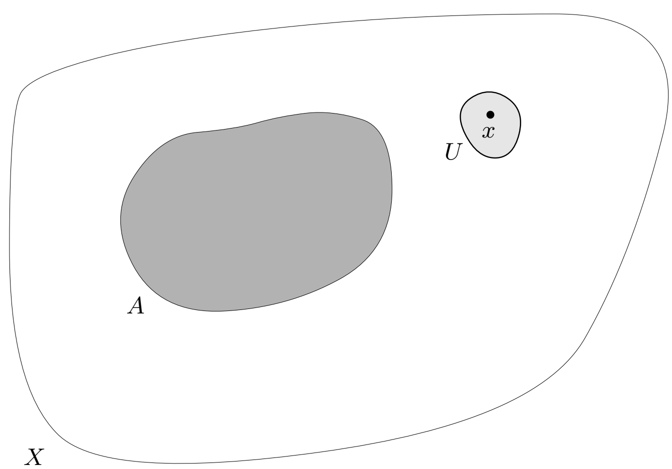


Figure 2.2 Isolated point x of a set A in a topological space X

In particular, an isolated point of a topological space X is a point that, as a singular set, is open in X.

2.3.7 Note. In a discrete space, every point belonging to any set is an isolated point of that set.

Opposed to the concept of isolated point, we have the following concept.

2.3.8 Definition. Let X be a topological space and let A be a subset of X. A point $x \in X$ is called a *cluster point* (or *point of accumulation*) of A, if every neighborhood V of x in X satisfies $(V - \{x\}) \cap A \neq \emptyset$.

2.3.9 Note. A cluster point x of A does not necessarily belong to A. We have the following assertion.

2.3.10 **Proposition**. *If x is a point in A, then either x is an isolated point or x is a cluster point of A.* □

2.3.11 Exercise. Prove that if x does not belong to A, but it is a cluster point of A, then x is a boundary point of A.

2.4 Neighborhood bases

Let X be a topological space. Given a neighborhood U of a point x in X, any set that contains U is also a neighborhood of x. Thus we can get to know all neighborhoods of x if we know an adequate collection of its neighborhoods –a collection containing "arbitrarily small" neighborhoods. For instance, an adequate collection might be that of its open neighborhoods.

2.4.1 Definition. Let X be a topological space and $x \in X$. A family $\mathcal{B}_x$ of neighborhoods of x in X is called a *neighborhood basis* of x in X if given any neighborhood U of x, there exists a neighborhood $V \in \mathcal{B}_x$ such that $V \subset U$. The elements of $\mathcal{B}_x$ are called *basic neighborhoods* of x.

For each point $x \in X$ one can give several neighborhood bases.

2.4.2 Examples. Let X be a topological space and take $x \in X$. The following are neighborhood bases:

(a) $\mathcal{B}_x = \{V \mid V \text{ is a neighborhood of } x \text{ in } X\}$.

(b) $\mathcal{B}_x = \{U \mid x \in U \text{ and } U \text{ is open } X\}$.

(c) $\mathcal{B}_x = \{\{x\}\}$, si $\{x\}$ is open in X (that is, if x is an isolated point of X).

(d) $\mathcal{B}_x = \{B_\varepsilon(x) \mid \varepsilon > 0\}$, if X is a metric space. (Even the family $\mathcal{B}'_x = \{B_{1/n}(x) \mid n \in \mathbb{N}\}$ is a neighborhood basis of x.)

Example (d) above suggests the following definition.

2.4.3 Definition. One says that a topological space X is *first-countable* if it *satisfies the first countability axiom*,, namely the axiom

(1-C) Each point of X has a countable neighborhood basis.

This "countability" axiom plays a very important role in convergence problems, as we shall see below.

The family $\mathcal{B}'_x$ of 2.4.2(d) proves the following.

2.4.4 **Theorem**. *Every metric (metrizable) space is first-countable.* □

In order to define a topology on a set, it is frequently useful to start with a family of what is going to be a collection of basic neighborhoods around each point. The desired topology should have the given family as a neighborhood basis around each point. We start with the next.

2.4.5 Definition. In a set X we say that a collection of families $\{\mathcal{B}_x\}_{x\in X}$ of subsets of X is a *neighborhood basis* if the following axioms hold:

(BN1) $V_i \in \mathcal{B}_x$, $i \in \mathcal{J}$, $\mathcal{J}$ finite $\Rightarrow \exists\, V \in \mathcal{B}_x$ such that $V \subset \bigcap_{i\in\mathcal{J}} V_i$;

(BN2) $V \in \mathcal{B}_x \Rightarrow x \in V$;

(BN3) $U \in \mathcal{B}_x \Rightarrow \exists\, V \in \mathcal{B}_x$ such that $\forall\, y \in V$ there exists $V_y \in \mathcal{B}_y$, for which $V_y \subset U$.

2.4.6 Exercise. Let X be a set and take a neighborhood basis $\{\mathcal{B}_x\}_{x\in X}$ as defined in 2.4.5. Prove that X admits a unique topology $\mathcal{A}$ determined by the neighborhood basis, for which each of the families $\mathcal{B}_x$ is a neighborhood basis (in the sense of Definition 2.4.1). (*Hint:* $A \in \mathcal{A}$ if and only if for every $x \in A$ there exists $V \in \mathcal{B}_x$ such that $V \subset A$. Alternatively, using $\{\mathcal{B}_x\}_{x\in X}$, one may define a system o neighborhoods in the sense of Definition 2.1.10.)

2.4.7 Exercise. At each point $x \in \mathbb{R}$ take

$$\mathcal{B}_x = \{(x-\varepsilon, x+\varepsilon) \mid \varepsilon > 0\}\,, \quad \text{si} \;\; x \neq 0\,, \quad \text{y}$$

$$\mathcal{B}_0 = \{(-\varepsilon,\varepsilon) \cup (-\infty,-n) \cup (n,\infty) \mid \varepsilon > 0, n \in \mathbb{N}\}\,.$$

(a) Prove that $\mathcal{B}_x$ is a neighborhood basis for each $x \in \mathbb{R}$.

(b) Describe the closed sets and the closure operator in the topology determined by these neighborhood bases.

The space L obtained from $\mathbb{R}$ with this topology is called *wrapped line*.

2.5 Continuity

2.5.1 Definition. Let X and Y be metric spaces with metrics d and d', respectively. A function $f : X \longrightarrow Y$ is said to be *continuous at a point* $x \in X$ if given $\varepsilon > 0$ there exists $\delta > 0$such that, if $d(y, x) < \delta$, then $d'(f(y), f(x)) < \varepsilon$. One says that f is *continuous* if it is continuous at each point $x \in X$.

The concept of continuity in topological spaces is, perhaps, the central concept of topology. Before introducing the general concept of continuity in topological spaces, it is convenient to state the concept in the case of metric spaces, which is an immediate generalization of the continuity in Euclidean spaces studied in elementary calculus courses.

This means, in common language, that a continuous map preserves the "nearness", in other words, it maps close points to close points.

Convention. In what follows, if it is not specified otherwise, the word **map** will mean a continuous function.

2.5.2 Examples.

(a) Any *constant map* $\kappa_{y_0} : X \longrightarrow Y$, $\kappa_{y_0}(x) = y_0$, is clearly continuous.

(b) The *identity map* $\mathrm{id}_X : X \longrightarrow X$ is continuous.

(c) The map $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

is not continuous at $t = 0$. Clearly f does not map points close to 0 to points close to $f(0) = 1$. Namely, no matter how close $t < 0$ is to 0, we have $d(f(t), f(0)) = d(0, 1) = |0 - 1| = 1$.

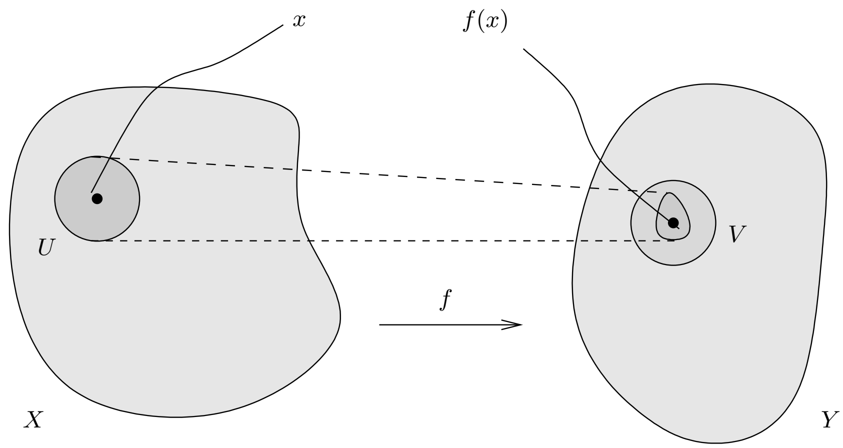


Figure 2.3 A continuous map sends neighborhoods into given neighborhoods

Given a map $f : X \longrightarrow Y$ between metric spaces and a point $x \in X$, we can rewrite the definition of continuity at x as follows.

2.5.3 **Proposition**. *A map $f : X \longrightarrow Y$ is continuous at x if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Equivalently, the map f is continuous at x if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$.* □

The last part of this assertion states, in other words, that f *is continuous if and only if, for every $\varepsilon > 0$, $f^{-1}(B_\varepsilon(f(x)))$ is a neighborhood of x.* This suggests how to formulate the general definition of a continuous map in arbitrary topological spaces.

2.5.4 DEFINITION. Let X and Y be topological spaces and take $f : X \longrightarrow Y$. We say that f is *continuous at a point $x \in X$*, if, given a neighborhood V of $f(x)$ in Y, the inverse image $f^{-1}(V)$ is a neighborhood of x in X. f is said to be *continuous* if it is continuous at any point in X.

2.5.5 REMARK. For metrizable topological spaces, this definition of continuity is equivalent to the definition of continuity in metric spaces, as shown in Proposition 2.5.3.

We also have the following characterization.

2.5.6 **Proposition**. *A map $f : X \longrightarrow Y$ is continuous at $x \in X$ if and only if, given a neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.* □

2.5.7 **Theorem**. *Let X, Y, and Z be topological spaces and let $f : X \longrightarrow Y$ be continuous at x and $g : Y \longrightarrow Z$ be continuous at $f(x)$. Then the composite $g \circ f : X \longrightarrow Z$ is continuous at x. Consequently, if f and g are continuous, then the composite $g \circ f$ is continuous.*

Proof: Take a neighborhood W of $gf(x) = g(f(x))$ in Z. Since g is continuous at $f(x)$, the inverse image $g^{-1}(W)$ is a neighborhood of $f(x)$ in Y. Analogously, since f is continuous at x, the inverse image $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is a neighborhood of x in X. $\square$

As we have seen, the topology of a space can be equivalently defined in several ways. Either giving its open sets, its closed sets, its interior operator, or its closure operator among others. Similarly, continuity can be characterized using any of these concepts. We have the following.

2.5.8 **Theorem**. *Let X and Y be topological spaces and take a map $f : X \longrightarrow Y$. The following are equivalent:*

(a) *f is continuous.*

(b) *For any open set $B \subseteq Y$, the inverse image $f^{-1}(B) \subseteq X$ is an open set.*

(c) *For any set $B \subseteq Y$, $f^{-1}(B^\circ) \subseteq f^{-1}(B)^\circ$.*

(d) *For any set $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.*

(e) *For any closed set $B \subseteq Y$, the inverse image $f^{-1}(B) \subseteq X$ is a closed set.*

Proof:

(a)⇒(b) Let B be open in Y and take $x \in f^{-1}(B)$. Thus $f(x) \in B$ and since B is open in Y, B is a neighborhood of $f(x)$. Since f is continuous (at x), $f^{-1}(B)$ is a neighborhood of x. Hence, since x is arbitrary, $f^{-1}(B)$ is open.

(b)⇒(c) Take a set $B \subseteq Y$. By (b), $f^{-1}(B^\circ)$ is open and is a subset of $f^{-1}(B)$. Therefore, it is also a subset of $f^{-1}(B)^\circ$.

(c)⇒(d) Take a set $A \subset X$. By (c), $f^{-1}((Y - f(A))^\circ) \subset f^{-1}(Y - f(A))^\circ$. Hence $X - f^{-1}\overline{f(A)} = f^{-1}(Y - \overline{f(A)}) = f^{-1}((Y - f(A))^\circ) \subset f^{-1}(Y - f(A))^\circ = (X - f^{-1}f(A))^\circ = X - \overline{f^{-1}f(A)}$. Therefore, since $A \subset f^{-1}f(A)$, one has $\overline{A} \subset \overline{f^{-1}f(A)} \subset f^{-1}\overline{f(A)}$. Taking the image under f of the first and the last term of the previous series we obtain $f(\overline{A}) \subset f(f^{-1}\overline{f(A)}) \subset \overline{f(A)}$.

(d)⇒(e) Let B be closed in Y. By (d) and since $ff^{-1}(B) \subset B$, we have $f(\overline{f^{-1}(B)}) \subset \overline{ff^{-1}(B)} \subset \overline{B} = B$. If we take inverse images, then we obtain

$f^{-1}f(\overline{f^{-1}(B)}) \subset f^{-1}(B)$. But $\overline{f^{-1}(B)} \subset f^{-1}f(\overline{f^{-1}(B)})$, thus $\overline{f^{-1}(B)} \subset f^{-1}(B)$ and consequently $f^{-1}(B) = \overline{f^{-1}(B)}$ is closed.

(e)⇒(a) Take any point $x \in X$ and let V be a neighborhood of $f(x)$ in Y. Without loss of generality, we may assume that V is open. Hence $Y - V$ is closed and, by (e), we have that $f^{-1}(Y - V) = X - f^{-1}(V)$ is also closed. Therefore, $f^{-1}(V)$ is open and since it clearly contains x, it is a (open) neighborhood of x in X. Thus f is continuous at x and since x is arbitrary, f is continuous. □

2.5.9 EXERCISE. Let X be a topological space. Prove that the following are equivalent:

(a) X has the indiscrete topology.

(b) Given any topological space Y and any map $f : Y \longrightarrow X$, f is continuous.

2.5.10 EXERCISE. Let X be a topological space. Prove that the following are equivalent:

(a) X has the discrete topology.

(b) Given any topological space Y and any map $f : X \longrightarrow Y$, f is continuous.

2.5.11 EXERCISE. Let X and Y be topological spaces, each with with the cofinite topology (see 2.1.2(e)) and take $f : X \longrightarrow Y$. Give a necessary and sufficient condition (in terms of the finite subsets of X and Y) in order for f to be continuous.

2.5.12 EXERCISE. Using only definitions and results from this and the previous chapter, show that the two maps $\mathbb{R}^2 \longrightarrow \mathbb{R}$ that map (s, t) to $s + t$ and to st are continuous.

2.6 HOMEOMORPHISMS

A topological space is a set together with certain additional structure given by its open sets (or, equivalently, by its closed sets, its interior operator, or its closure operator). In algebra, for instance, the objets of study are also sets with another "algebraic" structure, which might be a group structure, a ring structure, etc. From the point of view of algebra, one does not distinguish between two objects, when they are isomorphic, that is, when there is a bijection between the two underlying sets which preserves the structure. Similarly, in the case of topological spaces one can speak of "isomorphic" spaces when there exists a bijection between their underlying sets which preserves the "topological structure".

2.6.1 Definition. Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be a map. We say that f is a *homeomorphism* if it is a bijective continuous map and its inverse map $f^{-1} : Y \longrightarrow X$ is also continuous. If such a homeomorphism exists, then we say that the spaces X and Y are *homeomorphic*, and we denote this fact by $f : X \xrightarrow{\approx} Y$ or, more simply, by $X \approx Y$ (some authors write $X \cong Y$).

As we already said, a homeomorphism is a bijection which preserves the topological structure. This we shall see now.

2.6.2 **Theorem**. *Let X and Y be topological spaces and $f : X \longrightarrow Y$ a map. The following statements are equivalent:*

(a) *f is a homeomorphism.*

(b) *f is bijective and $f(A)$ is open in Y if and only if A is open in X.*

(c) *f is bijective and $f(A)$ is closed in Y if and only if A is closed in X.*

Proof: It is enough to observe that for every subset A of X, the inverse image $f^{-1}(f(A)) = A$. With this fact the proof follows immediately from Theorem 2.5.8. □

Statement (b) above says that f not only establishes a bijection between the points of X and those of Y, but also between the open sets of X and the open sets of Y. Similarly, statement (c) says the same, but for the closed sets instead. It is in this sense that a homeomorphism establishes an equivalence not only between the points of the underlying sets, but also between their topological structures.

2.6.3 Remark. If f is continuous and bijective, then it is not necessarily a homeomorphism. For example, if X a discrete space with more than one element and Y is an indiscrete space with the same underlying set of X, then the identity map $X \longrightarrow Y$ is bijective and continuous, but it is obviously not a homeomorphism.

2.6.4 Exercise. Prove that the map $f : [0, 1) \longrightarrow \mathbb{S}^1$, given by $f(t) = \mathrm{e}^{2\pi \mathrm{i} t}$, is bijective and continuous, but it is not a homeomorphism. (*Hint:* The inverse map $\log : \mathbb{S}^1 \to [0, 1)$ is not continuous at $1 \in \mathbb{S}^1$, since the inverse image of an open neighborhood of $0 \in [0, 1)$ of the form $[0, \varepsilon)$ with $\varepsilon < 1$, is not open in $\mathbb{S}^1$.)

2.6.5 Examples.

(i) The identity of of a topological space X, id_X, is a homeomorphism.

(ii) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are homeomorphisms, then $g \circ f : X \longrightarrow Z$ is a homeomorphism.

(iii) Consider the pierced unit sphere $X = \mathbb{S}^n - \{N\} \subset \mathbb{R}^{n+1}$, where $N = (0, \ldots, 0, 1)$ is the *north pole*. The map

$$p : X \longrightarrow \mathbb{R}^n \tag{2.6.6}$$

given by

$$p(x) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right),$$

where $x = (x_1, x_2, \ldots, x_{n+1}) \in X$, is a homeomorphism whose inverse is given by

$$p^{-1}(y) = \left(\frac{2y_1}{|y|^2 + 1}, \ldots, \frac{2y_n}{|y|^2 + 1}, \frac{|y|^2 - 1}{|y|^2 + 1} \right),$$

where

$$y = (y_1, \ldots, y_n) \in \mathbb{R}^n .$$

The map p is the so-called *stereographic projection*.

2.6.7 REMARK. The relation between topological spaces given by a homeomorphism is clearly an equivalence relation.

2.6.8 EXAMPLES.

(i) Every affine, nonsingular transformation $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, which is given by $f(x) = x_0 + L(x)$, where L is a linear isomorphism, is a homeomorphism.

(ii) I^n es homeomorphic to $\mathbb{B}^n$.

2.6.9 EXERCISE. Show an explicit homeomorphism for Example 2.6.8(ii).

2.6.10 EXERCISE. Take $X = \{x, y\}$. Prove that $\mathcal{A} = \{X, \emptyset, \{x\}\}$ is a topology. The corresponding topological space is called *Sierpiński space*.

2.6.11 EXERCISE. Write explicitly all possible topologies in a three-element set $X = \{x, y, z\}$. Prove that among all of them, there are exactly nine such that the corresponding topological spaces are not homeomorphic. Determine which of them are comparable and in that case indicate which is finer.

2.6.12 EXERCISE. Show that in a metrizable space X the following statements hold:

(a) Each one-point set is the intersection of all its (open) neighborhoods.

(b) Each one-point set is a closed set.

In $\mathbb{R}^n$ no point is open. Hence (a) implies that the intersection of open sets is not always an open set.

2.6.13 Exercise. Let X be a discrete space and $A \subset X$. Describe the boundary of A.

2.6.14 Exercise. Let X be a topological space and take $A \subset X$. Show that the boundary of A, ∂A, is a closed set.

2.6.15 Exercise. Let X be a metric space and take $y \in X$. Show that the function $f : X \longrightarrow \mathbb{R}$ given by $f(x) = d(x, y)$ is continuous.

2.6.16 Exercise. A set $A \subset \mathbb{R}^n$ is called *convex* if for any two points $x_0, x_1 \in A$, the *line segment*

$$[x_0, x_1] = \{(1-t)x_0 + tx_1 \mid t \in I\}$$

is contained in A. Show that if $A \subset \mathbb{R}^n$ is a closed, bounded convex set, whose interior is nonempty, then A is homeomorphic to the n-ball $\mathbb{B}^n$.

Chapter 3 Comparison of topologies

We have already seen that a given a set can be furnished with different topologies. In this chapter we shall discuss the relationship among the various topologies on the same underlying set. We shall introduce the concepts of topologies that are coarser or finer than other topologies. We shall as well answer the question of finding the coarsest and the finest topologies of a given family of topologies. This will be used to introduce the concepts of basis and subbasis of a topology, which are families of open sets smaller than the topology, but which however determine the topology.

3.1 Comparison of topologies

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two topologies on a set X, it is possible that the open sets according to one of the topologies, say $\mathcal{A}_2$, are also open sets according to the other topology $\mathcal{A}_1$, that is, $\mathcal{A}_2 \subset \mathcal{A}_1$, or it might happen that the topologies cannot be compared. We have the following.

3.1.1 Definition. We say that a topology $\mathcal{A}_2$ is (*strictly*) *coarser* than a topology $\mathcal{A}_1$, or that $\mathcal{A}_1$ is (*strictly*) *finer* than $\mathcal{A}_2$, if $\mathcal{A}_2 \subset \mathcal{A}_1$ (and they are different). We also say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *comparable*, if either $\mathcal{A}_2 \subset \mathcal{A}_2$ or $\mathcal{A}_2 \subset \mathcal{A}_1$.

The motivation of this terminology has to do in some sense with the "size" of the open sets. The more open sets we have in X, the finer they must be in order to fit into X.

3.1.2 Examples.

1. The discrete topology on X is finer than any other topology on X. On the other hand, the indiscrete topology on X is coarser than any other topology on X. Moreover, if X has more than one point, then the discrete topology is strictly finer than the indiscrete topology on X, and correspondingly, the indiscrete topology is strictly coarser than the discrete topology on X.

2. If $X = \{x, y\}$, then we can consider the topologies $\mathcal{A}_1 = \{X, \emptyset, \{x\}\}$ and $\mathcal{A}_2 = \{X, \emptyset, \{y\}\}$. The topologies $\mathcal{A}_1$ and $\mathcal{A}_2$ are not comparable.

The following result is clear from the definition.

3.1.3 **Theorem**. *Call X_1 a set X furnished with a topology $\mathcal{A}_1$ and call X_2 the same set X with a topology $\mathcal{A}_2$. Then* $\mathrm{id} : X_1 \longrightarrow X_2$ *is continuous if and only if $\mathcal{A}_1$ is finer than $\mathcal{A}_2$.* □

3.1.4 **Theorem**. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be topologies on X. Then the following statements are equivalent:*

(a) *$\mathcal{A}_1$ is finer than $\mathcal{A}_2$, i.e. $\mathcal{A}_2 \subset \mathcal{A}_1$.*

(b) *For each $x \in X$, any neighborhood of x according to $\mathcal{A}_2$ is also a neighborhood according to $\mathcal{A}_1$, i.e. $\mathcal{N}_x^2 \subset \mathcal{N}_x^1$.*

(c) *Every closed set according to $\mathcal{A}_2$ is closed according to $\mathcal{A}_1$, i.e. $\mathcal{C}_2 \subset \mathcal{C}_1$.*

(d) *For any set $A \subseteq X$, the closure of A according to $\mathcal{A}_1$ is contained in the closure of A according to $\mathcal{A}_2$, i.e. $\overline{A}^1 \subset \overline{A}^2$.*

(e) *For any set $A \subseteq X$, the interior of A according to $\mathcal{A}_1$ contains the interior of A according to $\mathcal{A}_2$, i.e. $A^{\circ 2} \subset A^{\circ 1}$.*

Proof: By 3.1.3, statement (a) is equivalent to say that $\mathrm{id} : X_1 \longrightarrow X_2$ is continuous. Thus, the equivalence of statements (b), (c), (d) and (e) with (a) is an immediate consequence of Definition 2.5.4 and of Theorem 2.5.8. □

3.1.5 Remark. Two given topologies on a set might not be comparable. A simple example is the set $X = \{x, y\}$ with the topologies $\mathcal{A}_1 = \{X, \emptyset, \{x\}\}$ and $\mathcal{A}_2 = \{X, \emptyset, \{y\}\}$. On the other hand, the relation "finer than" is *transitive*, namely, if $\mathcal{A}_1$ is finer than $\mathcal{A}_2$ and $\mathcal{A}_2$ is finer that $\mathcal{A}_3$, then $\mathcal{A}_1$ is finer than $\mathcal{A}_3$ (i.e. $\mathcal{A}_1 \supset \mathcal{A}_2$, $\mathcal{A}_2 \supset \mathcal{A}_3 \Rightarrow \mathcal{A}_1 \supset \mathcal{A}_3$). One also has that if $\mathcal{A}_1$ is finer than $\mathcal{A}_2$ and $\mathcal{A}_2$ is finer than $\mathcal{A}_1$, then $\mathcal{A}_1 = \mathcal{A}_2$, (i.e. $\mathcal{A}_1 \supset \mathcal{A}_2$, $\mathcal{A}_2 \supset \mathcal{A}_1 \Rightarrow \mathcal{A}_1 = \mathcal{A}_2$). In other words we can say that the family of all topologies on a given set constitute a *partially ordered set*. See 7.3.1 below.

Given any two topologies on a set X, there is always one which is finer than each of the two, namely the discrete topology. There is another topology which is coarser than both, namely the indiscrete topology. One may ask the following question: Is there among all topologies in X finer than $\mathcal{A}_1$ and $\mathcal{A}_2$ a coarsest topology? Another question: Is there among all topologies in X coarser than $\mathcal{A}_1$ and $\mathcal{A}_2$ one which is the finest topology? In what follows we shall study these questions.

3.1.6 Exercise. Consider the set $X = \{x, y, z\}$ and take $\mathcal{A}_1 = \{\emptyset, \{x\}, \{x, y\}, X\}$ and $\mathcal{A}_2 = \{\emptyset, \{x\}, \{y, z\}, X\}$. Show that $\mathcal{A}_1$ and $\mathcal{A}_2$ are topologies. Find the coarsest topology that contains both of them. Find the finest topology contained in both of them.

3.2 Intersection of topologies

Assume given a a family $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ of topologies on a set X. Our goal is to find a topology $\mathcal{A}$ which is *maximal* with the property that its open sets are also open sets of $\mathcal{A}_\lambda$ for all λ. In other words, we wish to construct the finest topology which is coarser than all topologies $\mathcal{A}_\lambda$. The ideal candidate for that is the intersection of all members $\mathcal{A}_\lambda$ of the given family.

3.2.1 Definition. Let $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ be a family of topologies on the same set X. We define the *infimum* of the family, denoted by $\inf\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$, to be the maximal (finest) topology $\mathcal{A}$ among all topologies which are coarser than all topologies $\mathcal{A}_\lambda$.

We have the following.

3.2.2 **Proposition**. *Let $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ be a family of topologies on the same set X. Then*

$$\mathcal{A} = \bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda$$

is a topology. Clearly it is the finest topology among all topologies that are coarser than $\mathcal{A}_\lambda$. Hence

$$\inf\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\} = \bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda \,.$$

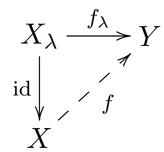
□

The following result provides us with a characterization of the infimum of a family of topologies. It has the form of a *universal property.*

3.2.3 **Theorem**. *Given a family $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ of topologies on a set X, its infimum $\mathcal{A}$ is characterized by the following universal property, given in two parts. Denote by X_λ the set X with the topology $\mathcal{A}_\lambda$ and by X the set X with the topology $\mathcal{A}$. Then*

(a) $\mathrm{id} : X_\lambda \longrightarrow X$ *is continuous for all* λ.

(b) *Given a function of sets* $f : X \longrightarrow Y$ *such that* $f_\lambda = f : X_\lambda \longrightarrow Y$ *is continuous for all* $\lambda \in \Lambda$, *then* $f : X \longrightarrow Y$ *is continuous. Put in a diagram*



$$f \text{ is continuous} \Leftrightarrow f_\lambda \text{ is continuous } \forall\, \lambda \in \Lambda.$$

Proof: (a) Since every open set A of X is open in X_λ, clearly $\mathrm{id} : X_\lambda \longrightarrow X$ is continuous for all $\lambda \in \Lambda$.

(b) If $f : X_\lambda \longrightarrow Y$ is continuous for all $\lambda \in \Lambda$, then for each open set B in Y, $f^{-1}(B)$ is in $\mathcal{A}_\lambda$ for all λ. Consequently $f^{-1}(B)$ lies in $\mathcal{A}$, and thus $f : X \longrightarrow Y$ is continuous.

Conversely, if the topology $\mathcal{A}$ has the property of the statement, and since $\mathrm{id} : X_\lambda \longrightarrow X$ is continuous, $\mathcal{A}$ lies in each $\mathcal{A}_\lambda$. Therefore $\mathcal{A}$ is contained in $\bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$. Let now X' be the set X with the topology of the intersection and take $Y = X'$ and $f = \mathrm{id}$, which clearly satisfy the property. Consequently $\mathrm{id} : X \longrightarrow X'$ is continuous and hence every open set of $\bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$ is an open set in $\mathcal{A}$. Thus $\mathcal{A} = \bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$. □

3.2.4 EXERCISE. Consider a set X and take $B \subseteq X$.

(a) Show that $\mathcal{A} = \{X, \emptyset, B\}$ is a topology on X. In particular, $\{\{a, b, c\}, \emptyset, \{b\}\}$ is a topology on the set $\{a, b, c\}$.

(b) Let $f : X \longrightarrow \{a, b, c\}$ be a function and assume $B = f^{-1}\{b\}$. Show that the topology $\mathcal{A}$ defined in (a) is the infimum of all topologies on X that make f continuous.

(c) Describe the interior operator and the closure operator for the topology $\mathcal{A}$ of (a).

3.3 SUPREMUM OF A FAMILY OF TOPOLOGIES

As before, assume given a a family $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ of topologies on the set X. Our question now is about the existence of a topology $\mathcal{A}$ which is *minimal* with the property that it contains all open sets of all the topologies $\mathcal{A}_\lambda$. In other words, the topology $\mathcal{A}$ must be the coarsest of all topologies that are finer than all the topologies of the family. The first that comes into mind is to consider the union $\bigcup_{\lambda\in\Lambda}\mathcal{A}_\lambda$, but of course we should ask us if this union is a topologie. Clearly, it is not.

3.3.1 Definition. Let $\{\mathcal{A}_\lambda \mid \lambda\in\Lambda\}$ be a family of topologies on the same set X. We define the *supremum* of the family, denoted by $\sup\{\mathcal{A}_\lambda \mid \lambda\in\Lambda\}$, to be the minimal (coarsest) topology $\mathcal{A}$ among all topologies which are finer than all topologies $\mathcal{A}_\lambda$.

3.3.2 Example. Take $X = \{x, y, z\}$ and the topologies $\mathcal{A}_1 = \{X, \emptyset, \{x\}\}$ and $\mathcal{A}_2 = \{X, \emptyset, \{y\}\}$ on X. Clearly the union $\mathcal{A}_1\cup\mathcal{A}_2$ is not a topology. However the only set missing in order to have a topology is $\{x, y\}$, namely the union of the only two nontrivial open sets in $\mathcal{A}_1$ and $\mathcal{A}_2$. Thus $\mathcal{A}_1\cup\mathcal{A}_2\cup\{\{x, y\}\}$ is the supremum of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Before we take in our hands the solution of the concrete problem we are facing, we shall analyze another more general problem. Let $\mathcal{F}$ be an arbitrary family of subsets of a set X. If $\mathcal{A}$ is a topology on X such that all elements of $\mathcal{F}$ are open sets of $\mathcal{A}$, then not only $\mathcal{F}\subset\mathcal{A}$, but $\mathcal{A}$ contains all finite intersections of elements of $\mathcal{F}$. Let $\mathcal{I}$ denote the family of all finite intersections of elements of $\mathcal{F}$. Hence we have

$$\mathcal{F}\subset\mathcal{I}\subset\mathcal{A}\,.$$

In particular, since X is the intersection of an "empty family", we have $X\in\mathcal{I}$.

The family $\mathcal{I}$ is not yet a topology; the desired topology $\mathcal{A}$ must also contain all unions of elements in $\mathcal{I}$. So take now the family $\mathcal{U}$ of all unions of elements of $\mathcal{I}$. In particular, $\emptyset\in\mathcal{U}$, since $\emptyset$ is the union of an "empty family." Thus we have

$$\mathcal{F}\subset\mathcal{I}\subset\mathcal{U}\subset\mathcal{A}\,.$$

3.3.3 **Proposition**. *The family $\mathcal{U}$ is a topology on X. Clearly $\mathcal{U}$ is the coarsest topology for which the elements $\mathcal{F}$ are open sets.* □

3.3.4 Definition. The family $\mathcal{U}$ is called the *topology generated* by $\mathcal{F}$. The original family $\mathcal{F}$ is called *subbasis* of the topology $\mathcal{U}$. The elements of $\mathcal{F}$ are called *subbasic open sets*.

Thus a subbasis $\mathcal{F}$ for a topology $\mathcal{U}$ is a family of open sets in $\mathcal{U}$ such that any open set in $\mathcal{U}$ is a union of finite intersections of open sets in $\mathcal{F}$.

Every topology $\mathcal{U}$ has at least one subbasis, namely $\mathcal{U}$ itself.

This concept of subbasis is quite useful. An example of this is the following.

3.3.5 **Theorem**. *Let X be topological space and $\mathcal{F}$ a subbasis of its topology. Take a map $f : Y \longrightarrow X$. Then f is continuous if and only if $f^{-1}(S)$ is an open set for all $S \in \mathcal{F}$.* □

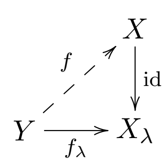
Now we may come back to the starting problem of this section. Given the family $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$, take $\mathcal{F} = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$. Thus the supremum of the family $\{\mathcal{A}_\lambda\}$ is the topology $\mathcal{U}$ determined by $\mathcal{F}$, namely, we have the following.

3.3.6 **Theorem**. *The supremum of a family $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ of topologies on a set X consists of arbitrary unions of finite intersections of open sets in $\mathcal{A}_\lambda$ for any λ.* □

3.3.7 **Theorem**. *The supremum of a family $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ of topologies on a set X is characterized by the following universal property, given in two parts. Denote by X_λ the set X with the topology $\mathcal{A}_\lambda$ and by X the set X with the topology $\mathcal{A}$. Then*

(a) $\mathrm{id} : X \longrightarrow X_\lambda$ *is continuous for all λ.*

(b) *Given a function of sets $f : Y \longrightarrow X$ such that $f_\lambda = f : Y \longrightarrow X_\lambda$ is continuous for all $\lambda \in \Lambda$, then $f : Y \longrightarrow X$ is continuous. Put in a diagram*



$$f \text{ is continuous} \Leftrightarrow f_\lambda \text{ is continuous } \forall\, \lambda \in \Lambda.$$

□

3.3.8 Exercise. Using 3.3.5, show in the same spirit as 3.2.3, the previous theorem.

3.3.9 Exercise. Show that the open half lines

$$(-\infty, b) \quad \text{y} \quad (a, +\infty)\,, \quad a, b \in \mathbb{R}\,,$$

constitute a subbasis of the usual topology of $\mathbb{R}$.

3.3.10 Exercise. Let X be an (infinite) set. Show that the family $\mathcal{S} = \{X - \{x\} \mid x \in X\}$ is a subbasis for the cofinite topology on X (2.1.2(e)).

3.3.11 Exercise. Let X be a set and let K_λ be a topological space, $\lambda \in \Lambda$. Consider a family of functions $\{f_\lambda : K_\lambda \longrightarrow X \mid \lambda \in \Lambda\}$ and put $\mathcal{A}_\lambda = \{A \subseteq X \mid f_\lambda^{-1} A \subseteq K_\lambda \text{ is open}\}$.

(a) Show that $\mathcal{A}_\lambda$ is a topology on X. Call X_λ the set X furnished with this topology.

(b) Let $\mathcal{A}$ be the infimum of the family of topologies $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$. Show that $\mathcal{A}$ is the supremum of all topologies on X that make $f_\lambda : K_\lambda \longrightarrow X$ continuous for all $\lambda \in \Lambda$.

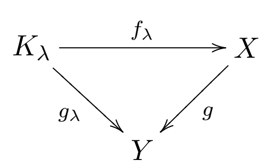
(c) Let $g : X \longrightarrow Y$ be a function, where X denotes the ste X furnished with the topology $\mathcal{A}$ of (b), and let Y be an arbitrary topological space. Show that g is continuous if and only if $g \circ f_\lambda : K_\lambda \longrightarrow Y$ is continuous for all $\lambda \in \Lambda$.

(d) For any two elements $\lambda, \mu \in \Lambda$ let $\varphi_\mu^\lambda : K_\lambda \longrightarrow K_\mu$ be a continuous map such that the equality $f_\mu \circ \varphi_\mu^\lambda = \varphi_\lambda$ holds. Assume that the maps φ_μ^λ have the following two properties:

   (i) Given $\lambda \in \Lambda$, the map $\varphi_\lambda^\lambda : K_\lambda \longrightarrow K_\lambda$ is the identity.

   (ii) Given $\lambda, \mu, \nu \in \Lambda$, one has the equality $\varphi_\nu^\lambda = \varphi_\nu^\mu \circ \varphi_\mu^\lambda : K_\lambda \longrightarrow K_\mu \longrightarrow K_\nu$.

   For each $\lambda \in \Lambda$, take a function $g_\lambda : K_\lambda \longrightarrow Y$ such that $g_\mu \circ f_\mu^\lambda = g_\lambda$, and let $g : X \longrightarrow Y$ satisfy $g \circ \varphi_\lambda = g_\lambda$. Show that g is continuous if and only if g_λ is continuous for each $\lambda \in \Lambda$. In a diagram



$$g \text{ is continuous } \Leftrightarrow g_\lambda \text{ is continuous for all } \lambda \in \Lambda\,.$$

3.4 Basis of a topology

In a metric or pseudometric space, balls are the building blocks that form all open sets, namely, any open set is a union of balls. As we already saw in the previous section, in general the open sets of a topological space are unions of finite intersections of open sets in a subbasis of the topology. This brings us to the following.

3.4.1 Definition. A family $\mathcal{B}$ of open sets in a topological space X is called a *basis* of (the topology of) X if every open set in X is a union of elements of the family $\mathcal{B}$. The elements of $\mathcal{B}$ are called *basic open sets*.

In particular, every basis of a topology is a subbasis.

We shall use below the following notion.

3.4.2 Definition. Let M be a set and $\leq$ a relation on M. If the pair $(M, \leq)$ satisfies the following axioms

(OR1) $a \leq a$ (the relation is *reflexive*).

(OR2) $a \leq b,\ b \leq c \Longrightarrow a \leq c$ (the relation is *transitive*).

(OR3) $a \leq b,\ b \leq a \Longrightarrow a = b$ (the relation is *antisymmetric*).

(OR4) For all $a, b \in M$, either $a \leq b$ or $b \leq a$.

then we say that the relation $\leq$ is a *total order* in M and that in that case, M is a *totally ordered set*. If $a \leq b$ and $a \neq b$, then we write $a < b$.

3.4.3 Examples.

(a) If X is a metric or pseudometric spave, then both

$$\mathcal{B} = \{B_\varepsilon(x) \mid \varepsilon > 0,\ x \in X\}$$

and

$$\mathcal{B}' = \{B_{1/n}(x) \mid n \in \mathbb{N}, x \in X\}$$

are bases for the topology of X.

(b) If $\mathcal{S}$ is a subbasis for a topology on X, then

$$\mathcal{B} = \{S_1 \cap \cdots \cap S_k \mid S_1, \ldots, S_k \in \mathcal{S}\}$$

is a basis for that topology.

(c) In $\mathbb{R}$, the open intervals constitute a basis for the usual topology.

(d) If X is a totally ordered set with order $\leq$ (see 3.4.2), then the *open intervals* $(a,b) = \{x \mid a \leq x \leq b,\, a \neq x \neq b\}$, $a,b \in X$, together with the *open rays* $(-\infty,b) = \{x \in x < b\}$ and $(a,\infty) = \{x \in X \mid a < x\}$, constitute a basis for a topology, called *order topology*. This topology has as a subbasis the open rays $(-\infty,b)$ and (a,∞), for all $a,b \in X$. Since by (c), the usual topology of $\mathbb{R}$ has the open intervals as basis, this topology is an order topology associated to the usual order of the reals.

3.4.4 Exercise. Consider the set $\mathbb{R}^2$ and define on it the *lexicographic order*, namely the order given by

$$(x,y) \leq (x',y') \quad \text{if either} \quad x \leq x' \quad \text{or if} \quad x = x' \quad \text{and} \quad y \leq y' \,.$$

Show that this is a total order. Furthermore, describe the open intervals and the open rays, and describe the corresponding order topology.

3.4.5 Exercise. The natural numbers $\mathbb{N}$ are an ordered set with minimal element. Describe the order topology of $\mathbb{N}$.

3.4.6 Exercise. Let $X = \{1,2\} \times \mathbb{N}$ have the lexicographic order. Then X becomes an ordered set with minimal element. Describe the order topology of X. Is it discrete?

3.4.7 **Proposition**. *Let X be a topological space and $\mathcal{B}$ a family of open sets in X. Then $\mathcal{B}$ is a basis for the topology of X if and only if, for every open set $U \subset X$, and for every $x \in U$, there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$.* □

3.4.8 Exercise. Show that in $\mathbb{R}^n$ the family $\{B_{1/n}(q) \mid n \in \mathbb{N},\, q \in \mathbb{Q}^n\}$ is a basis for the usual topology.

This last exercise shows that the topology of $\mathbb{R}^n$ admits a countable basis. This property is not shared by all topological spaces. An example would be any discrete uncountable space, since necessarily all its one-point subsets must belong to any basis. The property of having a countable basis is important as we shall see.

3.4.9 Definition. A topological space X is said to be *second-countable* if it *satisfies the second countability axiom*, namely the axiom

(2-C) The topology of X admits a countable basis.

3.4.10 **Proposition**. *If a topological space is second-countable, then it is also first-countable.*

Proof: If X satisfies the second countability axiom, then it has a countable basis $\mathcal{B}$. Take $x \in X$ and $\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in B\}$. Then $\mathcal{B}_x$ is a countable neighborhood basis at x. Hence X satisfies the first countability axiom. □

By Exercise 3.4.8, we have the following result.

3.4.11 **Theorem**. $\mathbb{R}^n$ *is second-countable.* □

3.4.12 DEFINITION. Let X be a topological space and take a subset $A \subset X$. We say that A is *dense* in X if the closure $\overline{A} = X$. A space X is called *separable* if it contains a countable dense subset.

3.4.13 EXERCISE. Let $\mathbb{R}_l$ denote $\mathbb{R}$ with the so-called *lower half-open interval topology* or *lower limit topology* that has as basis the intervals $[a, b)$, $a < b$. $\mathbb{R}_l$ is also known as the *Sorgenfrey line* and is also denoted by $\mathbb{E}$. Similarly, $\mathbb{R}_u$ denotes $\mathbb{R}$ with the *upper half-open interval topology* or *upper limit topology*, which has as basis the half-open intervals $(a, b]$, $a < b$.

(a) For a subset $A \subseteq \mathbb{R}_l$ show that a point x lies in the closure of A if and only if there is a sequence (x_n) in A such that $x_n \geq x$ and $|x_n - x| \to 0$. What is the corresponding statement for $\mathbb{R}_u$?

(b) Show that a function $f : \mathbb{R}_l \longrightarrow \mathbb{R}$ (with the usual topology on $\mathbb{R}$) is continuous if and only if it is continuous from the right at each point x, that is, $\lim f(x + \varepsilon) = f(x)$ where the limit is as $\varepsilon \to 0$ with $\varepsilon > 0$. What is the corresponding statement for $\mathbb{R}_u$?

3.4.14 EXERCISE. Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

(a) Assume that f is *continuous from the right*, namely, $\lim_{x \to a^+} f(x) = f(a)$ for each $a \in \mathbb{R}$, where $x \to a^+$ means that the limit is taken if $x - a > 0$. Prove that f is continuous when considered as a function $f : \mathbb{R}_l \longrightarrow \mathbb{R}$.

(b) What can be said about the continuity of f when considered as $f : \mathbb{R} \longrightarrow \mathbb{R}_l$ and $f : \mathbb{R}_l \longrightarrow \mathbb{R}_l$?

3.4.15 EXERCISE. Let Y be an ordered set furnished with the order topology and assume that $f, g : X \longrightarrow Y$ are continuous.

(a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \longrightarrow Y$ be the function given by

$$h(x) = \min\{f(x), g(x)\}\,.$$

Show that h is continuous.

3.4.16 Example. More generally than 3.4.3(c), if we have a finite family of ordered sets $A_1, \dots, A_n$, with order relations $<_1, \dots, <_n$, respectively, we can define a *lexicographic order* relation in the product $A_1 \times \cdots \times A_m$ as follows: $(a_1, \dots, a_n) < (a'_1, \dots, a'_n)$ if and only if either $a_1 <_1 a'_1$, or $a_1 = a'_1, \dots, a_m = a'_m$ and $a_{m+1} <_{m+1} a'_{m+1}$ for some $m < n$. This is a source of interesting examples of topological spaces.

A useful example is the so-called *long line* $\mathbb{L}$, which is a topological space analogous to the real line, but much longer. Since it behaves locally just like the real line, but has different large-scale properties, it serves as an important counterexample in topology, see [17]. To construct it, take the first uncountable ordinal ω_1 and its product with the semiclosed interval $[0, 1)$ and delete the smallest element. Then equip this product with the order topology that arises from the lexicographic order (considering the usual order relation on $[0, 1)$, which defines the usual topology). This long line $\mathbb{L}$ can be intuitively described as uncountably many copies of $[0, 1)$ put together one after the other, according to the order of the first uncountable ordinal. The long line is locally the same as $\mathbb{R}$, since the neighborhoods of a point in $\mathbb{L}$ are essentially standard intervals of the form (a, b), just as in $\mathbb{R}$. However, a basis for its topology must be the union of bases on each copy of $[0, 1)$. Because there are uncountably many copies of $[0, 1)$ it is impossible to choose a countable basis for the topology of $\mathbb{L}$. Briefly, $\mathbb{L}$ is a first-countable space which is not second-countable. As a consequence, $\mathbb{L}$ cannot be embedded (see 4.1.19) as a subspace of $\mathbb{R}^n$ for any n, because otherwise it would be second-countable.

We have, however, the following result, whose proof is quite simple.

3.4.17 **Proposition**. *Every separable metric space is second countable.*

Proof: Let X be a separable metric space and take a countable dense subset $A \subset X$. Consider the countable family $\mathcal{B} = \{B_{1/n}(a) \mid n \in \mathbb{N}, a \in A\}$. We shall see that $\mathcal{B}$ is a basis for the topology of X. To see this, let $U \subset X$ be open and take $x \in U$. Let $\varepsilon > 0$ be such that $B_\varepsilon(x) \subset U$. Take $n \in \mathbb{N}$ such that $1/n < \varepsilon/2$. Since A is dense, we can take $a \in A$ such that $d(x, a) < 1/n$. Then $x \in B_{1/n}(a) \subset B_\varepsilon(x) \subset U$. Consequently $\mathcal{B}$ is a basis for the topology. □

In opposition to Example 3.4.19 one may prove the following.

3.4.18 EXERCISE. Show that every second-countable space X is separable and first-countable. (*Hint:* By 3.4.10, X is first-countable. If $\mathcal{B} = \{B_n\}$ is a countable basis, use the axiom of choice to pick an element $x_n \in B_n$. The set $\{x_n\}$ is countable and dense in X.)

Exercise 3.4.8 suggests that one might similarly prove the converse of the statement in the previous exercise, namely, that a separable and first-countable space is second-countable. However this is not the case as described next.

3.4.19 EXAMPLE. The Sorgenfrey line $\mathbb{E}$ defined in 3.4.13 is a separable space, since the subset of rational numbers is clearly dense. $\mathbb{E}$ is also first-countable, since the intervals $[x, x + 1/n)$ constitute a countable neighborhood basis around x. However $\mathbb{E}$ is not second-countable. A proof of this fact can be found in [20, 16.12].

3.4.20 EXERCISE. Show that a family $\mathcal{B}$ of subsets of a space X is a basis for the topology, for which it is a subbasis, if and only if X is union of all the elements of $\mathcal{B}$ and the finite intersections of the elements of $\mathcal{B}$ are unions of elements of $\mathcal{B}$.

3.4.21 EXERCISE. Recall the metric space of sequences

$$\ell^2 = \{(x_n) \mid x_n \in \mathbb{R},\ \sum x_n^2 < \infty\}$$

with the metric

$$d((x_n), (y_n)) = \sqrt{\sum (x_n - y_n)^2}\,.$$

Show that ℓ^2 is second-countable. (*Hint:* Consider the almost-null rational sequences.)

3.4.22 EXERCISE. Is Proposition 3.4.17 true for pseudometric spaces?

3.4.23 EXERCISE. Show that every open set in $\mathbb{R}$ is the union of a collection of disjoint open intervals (a, b) where one allows $a = -\infty$ and $b = \infty$. That is, this collection is a basis for the usual topology.

3.4.24 EXERCISE. Consider the set $K \subset \mathbb{R}$ consisting of all numbers of the form $\frac{1}{n}$, for $n \in \mathbb{N}$, and let $\mathcal{B}$ be the collection of all open intervals (a, b) along with all sets of the form $(a, b) - K$.

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. Denote the resulting space by $\mathbb{R}_K$.

(b) Show that the topologies of $\mathbb{R}_l$ (see the exercise above) and of $\mathbb{R}_K$ are strictly finer than the usual topology on $\mathbb{R}$, but they are not comparable with one another.

(c) Show that $\mathbb{R}_l$ and $\mathbb{R}_u$ are homeomorphic.

3.4.25 Exercise. Consider the following topologies on $\mathbb{R}$:

$\mathcal{A}_1 =$ the usual topology,

$\mathcal{A}_2 =$ the topology of $\mathbb{R}_K$,

$\mathcal{A}_3 =$ the cofinite topology,

$\mathcal{A}_4 =$ the upper limit topology of $\mathbb{R}_u$,

$\mathcal{A}_5 =$ the topology that has all sets $(-\infty, b) = \{t \in \mathbb{R} \mid t < b\}$ as basis.

Compare all these topologies (namely, determine for each of them, which of the others is finer or coarser).

To finish this section, and with it also the chapter, we have the following lemma that will be useful in Chapter 9.

3.4.26 **Lemma**. *If a topological space X is second-countable, then every basis of X contains a countable basis.*

Proof: Let $\mathcal{Q} = \{Q_n\}$ be a countable basis for the topology of X and let $\mathcal{B} = \{B_\lambda\}$ be an arbitrary basis. Each Q_m is a union of open sets B_λ, namely, if $x \in Q_m$, then there exists $B_\lambda \in \mathcal{B}$ such that $x \in B_\lambda \subset Q_m$. Analogously, if $x \in B_\lambda$, then there exists Q_n such that $x \in Q_n \subset B_\lambda$. I.e.

$$x \in Q_n \subset B_\lambda \subset Q_m.$$

Now consider the pairs (Q_n, Q_m) such that there exists B_λ in such a way that $Q_n \subset B_\lambda \subset Q_m$ and for each of these pairs denote by $B_{n,m}$ one such B_λ. The set $\{B_{n,m}\}$ is a countable subset that clearly is a basis, as we wanted. □

Chapter 4 Generating topological spaces

We shall see now that given any topological spaces, there are various constructions that allow to obtain new topological spaces from the given ones. We shall thus analyze the topology induced on subsets, namely the relative topology on subspaces, as well as the topology coinduced on a quotient, namely the quotient (or identification) topology on quotient spaces. We shall also study the topology in a product and in a disjoint union, namely the topological product and the topological sum of spaces.

4.1 Induced topology

In this section we shall start by considering subsets of a topological space as topological spaces by themselves. We shall describe their topology in order that the inclusion maps become continuous. More generally, if X is a topological space and X' is a set, then we shall consider a function $h : X' \longrightarrow X$ and answer the question about the coarsest topology in X' such that h is continuous.

4.1.1 Definition. Let X be a topological space and take $X' \subset X$. If $\mathcal{A}$ is the topology in X, then $\mathcal{A}' = \{A \cap X' \mid A \in \mathcal{A}\}$ is a topology. This topology is called the *relative topology* induced on X' by X. The space X' with this topology is called (topological) *subspace* of X.

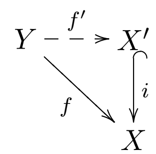
4.1.2 Exercise. Prove that $\mathcal{A}'$ is in fact a topology. Moreover, prove that it is the coarsest topology that makes the inclusion map $i : X' \hookrightarrow X$ continuous.

4.1.3 Exercise. Let X be a topological space and consider $X'' \subset X' \subset X$. Assume that X' has the relative topology induced by X and that X'' has the relative topology induced by X'. Show that X'' has the relative topology induced by X.

4.1.4 **Proposition**. *Let X be a topological space and take $X' \subset X$. Then the relative topology induced by X on X' is characterized by the following universal property:*

(i) *The inclusion* $i : X' \hookrightarrow X$ *is continuous.*

(ii) *A map* $f' : Y \longrightarrow X'$ *is continuous if and only if the composite* $f = i \circ f'$ *is continuous. In a diagram*



f *is continuous* $\Leftrightarrow$ f' *is continuous.*

Proof:

(i) This is clear since for every open set $A \subset X$, the inverse image $i^{-1}(A) = A \cap X'$ is open in X' by definition.

(ii) If f' is continuous, then by (i), $f = i \circ f'$ is continuous. Conversely, assume that f is continuous. Take an open set in $A' \subset X'$. By definition $A' = A \cap X'$, with $A \subset X$ open. Thus, since f is continuous, $f'^{-1}(A \cap X') = f'^{-1} i^{-1}(A) = f^{-1}(A)$ is open. Therefore f' is continuous. □

4.1.5 Example. The *Zariski topology* in an algebraic variety $V \subset \mathbb{R}^n$, defined in 2.2.23, is the relative topology of V induced by the Zariski topology in $\mathbb{R}^n$.

The concept of relative topology just defined is a special case of a more general concept that we are about to define.

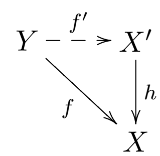
4.1.6 Definition. Let X' be a set and let X be a topological space. Take a function $h : X' \longrightarrow X$. If $\mathcal{A}$ is the topology in X, then $\mathcal{A}' = \{h^{-1}(A) \mid A \in \mathcal{A}\}$ is a topology on X' as one easily verifies. The topology $\mathcal{A}'$ is called the *topology on* X' *induced by* X *through* h or simply *topology induced through* h. This topology is the coarsest that makes h continuous.

In the same spirit as 4.1.4, we have the following theorem.

4.1.7 **Theorem**. *Let* X *be a topological space and let* $h : X' \longrightarrow X$ *be a function of sets. Then the topology induced on* X' *by* X *through* h *is characterized by the following universal property.*

(i) *The map* h *is continuous.*

(ii) *A function* $f' : Y \longrightarrow X'$ *is continuous if and only if* $f = h \circ f'$ *is continuous. In a diagram*



f *is continuous* $\Leftrightarrow$ f' *is continuous.*

The *proof* is essentially the same as that of 4.1.4 and we leave it as an *exercise* to the reader. □

4.1.8 REMARK. Taking $f' : X' \longrightarrow X'$ as the identity map, it is easy to see that in Theorems 4.1.4 and 4.1.7, assertion (ii) implies assertion (i). Therefore (i) is redundant. We write it explicitly because of the importance of the property. In this sense, the universal property that characterizes the induced topology is (ii) in both cases .

The next theorem provides other characterizations of the induced topology.

4.1.9 **Theorem**. *Let* X *and* X' *be topological spaces and let* $h : X' \longrightarrow X$ *be a continuous map such that* X' *has the topology induced through* h*. Then*

(a) $A' \subset X'$ *is open if and only if* $A' = h^{-1}(A)$ *for some* A *open in* X*.*

(b) $A' \subset X'$ *is closed if and only if* $A' = h^{-1}(A)$ *for some* A *closed in* X*.*

(c) $V' \subset X'$ *is a neighborhood of* x' *if and only if* $V' \supset h^{-1}(V)$ *for some neighborhood* V *of* $h(x')$ *in* X*.* □

4.1.10 EXERCISE. Take a map $h : X' \longrightarrow X$ and assume that X' has the topology induced through h. Show that if $\mathcal{B}$ is a basis for the topology of X, then $\{h^{-1}(B) \mid B \in \mathcal{B}\}$ is a basis for the topology of X'. Furthermore show that if $\mathcal{B}_{h(x')}$ is a neighborhood basis around $h(x')$ in X for some point $x' \in X'$, then $\{h^{-1}(V) \mid V \in \mathcal{B}_{h(x')}\}$ is a neighborhood basis around x' in X'.

4.1.11 REMARK. Let X and X' be topological spaces and let $h : X' \longrightarrow X$ be a continuous map such that X' has the topology induced through h. If X first-, resp. second-countable, then X' is first-, resp. second-countable. In other words, the two countability axioms are *hereditary*. This is clear since if $\mathcal{B}$ is a basis for the topology of X,resp. $\mathcal{B}_{h(x')}$ is a neighborhood basis around $h(x')$, then $\{h^{-1}(B) \mid B \in \mathcal{B}\}$ is a basis for X', resp. $\{h^{-1}(B) \mid B \in \mathcal{B}_{h(x')}\}$ is a neighborhood basis around x' (see Exercise 4.1.10).

4.1.12 **Theorem**. *Let X, Y, and Z be topological spaces and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be maps such that Y has the topology induced through g. If X has the topology induced through f, then it also has the topology induced through $g \circ f$.*

Proof: It is a straightforward application of the formula $(g \circ f)^{-1}(A) = f^{-1}g^{-1}(A)$. □

Since one of the important instances of induced topology is that of the relative topology of a subspace, it is convenient to point out some details in this respect.

4.1.13 Exercise. Rewrite 4.1.9-4.1.12 for the case of the relative topology in a subspace.

4.1.14 Definition. Let $A \subset X$ have the relative topology. In this case we call the continuous map $i : A \longrightarrow X$ simply the *inclusion* (and denote it frequently by $A \hookrightarrow X$). The open sets of A are called *relative open sets*; the closed sets of A are called *relative closed sets*.

Observe that A, not necessarily being open or closed in X, is a relative open and a relative closed set (in A).

4.1.15 **Proposition**.

(a) *The relative open sets in A are open in X if and only if A is open in X.*

(b) *The relative closed sets in A are closed in X if and only if A is closed in X.* □

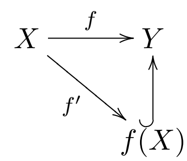
From here on, unless otherwise indicated, all subsets of a topological space will be considered as subspaces, namely, they will be furnished with the relative topology, and the word *inclusion* will always refer to that of a subspace into the given space. In particular, we have the following result.

4.1.16 **Proposition**. *Let X be a metric space and take $A \subset X$ with the induced metric. Then the topology in A given by the induced metric coincides with the relative topology.* □

4.1.17 Exercise. Let X be a (pseudo)metric space with (pseudo)metric d and let X' be a set. If $h : X' \longrightarrow X$ is a function and $x', y' \in X'$, define $d'(x', y') = d(h(x'), h(y'))$. Then d' is a pseudometric in X' (see Exercise 1.5.11). Show that the topology determined by the pseudometric in X' is the topology induced through h.

With this, we have very well determined what the topology of the subsets of $\mathbb{R}^n$ is, and in particular, the topology of all spaces given as a motivation in the Introduction and in Section 1.1.

4.1.18 **Theorem**. *Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be a function. Let $f(X)$ be the image of X under f with the relative topology and let $f' : X \longrightarrow f(X)$ be such that $f'(x) = f(x)$. Then f is continuous if and only if f' is continuous. In a diagram we have*



$$f \text{ is continuous} \Leftrightarrow f' \text{ is continuous.}$$ □

If in the previous theorem, f defines a homeomorphism with its image, that is, if f' is a homeomorphism, then we can see X as a subspace of Y. We have the next concept.

4.1.19 Definition. A continuous map $e : A \longrightarrow X$ is called an *embedding* if the restriction of e, $e' : A \longrightarrow e(A)$ given by $e'(a) = e(a)$, is a homeomorphism when furnishing $e(A)$ with the relative topology.

Thus, in particular, every inclusion is an embedding. We shall frequently extend the use of the word *inclusion* for an embedding that in some sense is canonical.

There are other important examples.

4.1.20 Examples.

1. The map $e : (0,1) \longrightarrow \mathbb{S}^1 \subset \mathbb{C}$ given by $e(t) = \mathrm{e}^{2\pi \mathrm{i} t}$ is an embedding, whose image is the complement of $1 \in \mathbb{S}^1$. However, the extended map $e' : [0,1) \longrightarrow \mathbb{S}^1$ given by the same formula, though still injective (even bijective), is not an embedding anymore (otherwise it would be a homeomorphism –*exercise*). Notice that there are homeomorphisms
$$h_1 : \mathbb{R} \longrightarrow (-1,1) \quad \text{and} \quad h_2 : (-1,1) \longrightarrow (0,1)$$
given by $h_1(r) = \frac{r}{1+|r|}$ and $h_2(s) = \frac{s+1}{2}$. The composite
$$e' : \mathbb{R} \xrightarrow{h_1} (-1,1) \xrightarrow{h_2} (0,1) \xrightarrow{e} \mathbb{S}^1$$
is an embedding of the real line into the circle (missing only 1).

2. The map $e : \mathbb{R}^n \longrightarrow \mathbb{S}^n$ given by

$$e(y) = \left(\frac{2y_1}{|y|^2+1}, \ldots, \frac{2y_n}{|y|^2+1}, \frac{|y|^2-1}{|y|^2+1} \right),$$

   where $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, is an embedding. Indeed, it is the inverse of the stereographic projection (see 2.6.6).

3. Put $D^n_+ = \{(x_1, \ldots, x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 1,\ x_{n+1} \geq 0\} \subset \mathbb{S}^n$. Then the projection $p : D^n_+ \longrightarrow \mathbb{R}^n$ such that $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$, is an embedding with image $\mathbb{D}^n \subset \mathbb{R}^n$.

4. The map $e : \mathbb{D}^n \longrightarrow \mathbb{S}^n$ from the unit n-ball into the n-sphere given by $e(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sqrt{1 - x_1^2 - \cdots - x_n^2})$, is an embedding whose image is $D^n_+ \subset \mathbb{S}^n$

4.1.21 DEFINITION. Let X and Y be topological spaces, $A \subset X$, $i : A \longrightarrow X$ the inclusion, and $f : X \longrightarrow Y$. The composite $f \circ i$ is called *restriction* of f to A, and is denoted by $f|_A$.

Clearly, if f is continuous, then $f|_A$ is continuous. However, the inverse need not be true.

4.1.22 EXAMPLE. Take $X = Y = \mathbb{R}$, $A = \mathbb{Q}$, and let $f : X \longrightarrow Y$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}\,, \\ 1 & \text{if } x \in \mathbb{R} - \mathbb{Q}\,. \end{cases}$$

Then $f|_A$ is continuous and $f|_{X-A}$ is continuous, but f is not continuous.

It is a quite frequent fact in topology to have maps that, as in the previous example, are piecewise defined, namely, they are given by different formulas on different portions of their definition domain. The next results give criteria to know when such maps are continuous.

4.1.23 **Theorem**. *Let $\{A_1, A_2, \ldots, A_k\}$ be a* closed cover *of X; that is, the sets A_i, $i = 1, \ldots, k$, are closed subsets of X such that $X = \bigcup_{i=1}^k A_i$. A map $f : X \longrightarrow Y$ is continuous if and only if the restriction $f_i = f|_{A_i}$ is continuous for every $i = 1, 2, \ldots, k$.*

Proof: If f is continuous, then clearly its restrictions f_i are continuous.

Conversely, let $G \subset Y$ be closed. Then for every i, $f_i^{-1}(G)$ is closed in A_i and therefore also in X. Since $f_i^{-1}(G) = f^{-1}(G) \cap A_i$, $f^{-1}(G) = \bigcup_{i=1}^k f_i^{-1}(G)$. This set is closed, because it is a finite union of closed subsets. □

The previous result can be stated as follows.

4.1.24 **Corollary**. *Let $\{A_1, A_2, \ldots, A_k\}$ be a closed cover of X and let $f_i : A_i \longrightarrow Y$, $i = 1, \ldots, k$, be continuous maps such that for every i, j, $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$. Then there exists a continuous map $f : X \longrightarrow Y$ such that $f|_{A_i} = f_i$ for all i.* □

4.1.25 **Theorem**. *Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be an* open cover *of X; that is, it consists of open subsets of X such that $X = \bigcup_{\lambda \in \Lambda} A_\lambda$. A map $f : X \longrightarrow Y$ is continuous if and only if the restriction $f_\lambda = f|_{A_\lambda}$ is continuous for every $\lambda \in \Lambda$.*

The *proof* is analogous to that of the previous theorem. □

The last theorem can be stated as follows.

4.1.26 **Corollary**. *Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be an open cover of X and let $f_\lambda : A_\lambda \longrightarrow Y$, $\lambda \in \Lambda$, be continuous maps such that for every $\lambda, \mu \in \Lambda$, $f_\lambda|_{A_\lambda \cap A_\mu} = f_\mu|_{A_\lambda \cap A_\mu}$. Then there exists a continuous map $f : X \longrightarrow Y$ such that $f|_{A_\lambda} = f_\lambda$ for all $\lambda \in \Lambda$.* □

4.1.27 EXERCISE. Prove the following statements:

(a) Every subspace of a discrete space is discrete.

(b) Every subspace of an indiscrete space is indiscrete.

4.1.28 EXERCISE. Prove that every finite subspace of a metric space is discrete. Observe, however, that the induced metric is not necessarily the discrete metric.

4.1.29 EXERCISE. Prove that the subsets $\mathbb{Z}$ and $\mathbb{N}$ of $\mathbb{R}$ are discrete spaces. More generally, every *lattice* in $\mathbb{R}^n$, that is, every subspace of the form $\{k_1a_1 + \cdots + k_na_n \mid k_1, \ldots, k_n \in \mathbb{Z}\}$, where $a_1, \ldots, a_n \in \mathbb{R}^n$ are linearly independent vectors, is a discrete space. Prove, more precisely, that any lattice in $\mathbb{R}^n$ is (canonically) homeomorphic to $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z} \subset \mathbb{R}^n$.

4.1.30 EXERCISE. Let X' be a set and assume that X is a topological space with topology $\mathcal{A}$. Take an arbitrary family of functions $\{h_\alpha : X' \longrightarrow X \mid \alpha \in \mathcal{I}\}$.

(a) Consider the topology $\mathcal{A}'$ in X' which has as a subbasis the family of sets $\{h_\alpha^{-1}(A) \mid A \in \mathcal{A}, \alpha \in \mathcal{I}\}$. Show that $\mathcal{A}'$ is the coarsest topology on X' that makes all functions h_α continuous. This is the so-called *topology in X' induced by X through the family $\{h_\alpha\}$.*

(b) Prove that the Zariski topology (2.2.23) is the topology on $\mathbb{R}^n$ induced by $\mathbb{R}$ furnished with the cofinite topology, through the family of polynomial functions.

4.2 Identification topology

Take a topological space X and a set X'. Assume that a function $f : X \longrightarrow X'$ is given. In this section we shall see which is the finest topology $\mathcal{A}'$ on X' for which f is continuous. Let us consider the family $\mathcal{F} = \{A' \subset X' \mid f^{-1}(A') \text{ is open en } X\}$. If f has to be continuous, then the open sets in the desired topology $\mathcal{A}'$ on X' should be contained in $\mathcal{F}$. Furthermore, this topology has to be as fine as possible. It would be ideal that precisely the family $\mathcal{F}$ is a topology. Then we may take $\mathcal{A}' = \mathcal{F}$. Indeed, it is straightforward to prove the following.

4.2.1 **Proposition**. *The family $\mathcal{F}$ is a topology, which we denote by $\mathcal{A}'$.* □

Notice that if x' does not belong to the image of f, then since $f^{-1}(\{x'\}) = \emptyset$ is open, the set $\{x'\}$ is open, that is, the complement $X' - f(X)$ is discrete. Therefore, the nontrivial part of the desired topology $\mathcal{A}'$ consists of sets in the image of f. Hence, to avoid carrying the trivial part, from now on (dually to the corresponding case in the previous section, where we assumed $A \subset X$) we shall always assume here that the map f is surjective, that is, $f(X) = X'$.

4.2.2 Definition. Let X be a topological space, Y a set, and let $f : X \longrightarrow Y$ be surjective. The finest topology in Y for which f is continuous (namely the topology for which B is open in Y if and only if $f^{-1}(B)$ is open in X) is called *identification topology* coinduced by f or, dually to 4.1.6, the *topology in Y coinduced by X through f* or simply *topology coinduced by f*). The map f is then called *identification* (dually to the term *inclusion*, some authors call an identification a *proclusion*).

This terminology is justified by thinking that the map f "identifies" in just one point all those points which have the same point as their image.

4.2.3 Exercise. Consider the surjective function $f : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ given by

$$f(t) = \begin{cases} -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0\,. \end{cases}$$

Describe the identification topology on the set $\{-1, 0, 1\}$.

It is immediate to prove the following result.

4.2.4 **Proposition**. *Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be surjective. The following are equivalent:*

(a) *f is an identification.*

(b) *A subset of Y is open if and only if its inverse image under f is open in X.*

(c) *A subset of Y is closed if and only if its inverse image under f is closed in X.* □

The identifications have the following nice behavior.

4.2.5 **Proposition**. *Assume that X, Y, and Z are topological spaces.*

(a) $\mathrm{id}_X : X \longrightarrow X$ *is an identification.*

(b) *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are identifications, then $g \circ f : X \longrightarrow Z$ is an identification.* □

4.2.6 EXERCISE. Conversely to (b) above, show that if $g \circ f : X \longrightarrow Z$ is an identification, then g is an identification.

Conversely to (a) above one has the following.

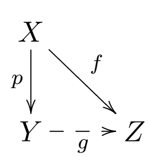
4.2.7 **Proposition**. *Let $f : X \longrightarrow Y$ be bijective. Then f is an identification if and only if it is a homeomorphism.* □

Dually to 4.1.7, we have the following universal property that characterizes the concept of identification.

4.2.8 **Theorem**. *Let X and Y be topological spaces and let $p : X \longrightarrow Y$ be a surjective function. Then p is an identification if and only if the following conditions hold:*

(i) *The map p is continuous.*

(ii) *A map $g : Y \longrightarrow Z$ is continuous if and only if the composite $f = g \circ p$ is continuous. In a diagram*



f is continuous $\Leftrightarrow$ g is continuous.

Proof: (i) is clear. To prove (ii), assume first that p is an identification. Hence p is continuous, and if g is continuous, so is $g \circ p = f$.

Conversely, suppose that f is continuous. To see that g is continuous, let $C \subseteq Z$ be closed. We prove that $g^{-1}(C) \subseteq Y$ is closed. But this is clear since $p^{-1}(g^{-1}(C)) = (g \circ p)^{-1}(C) = f^{-1}(C) \subseteq X$ is closed, because f is continuous. Hence condition (ii) holds.

Now assume that conditions (i) and (ii) hold and we prove that p is an identification. To see this, consider the identification space Y' obtained by furnishing Y with the identification topology coinduced by p. Call $p' : X \longrightarrow Y'$ the identification. Consider the identity, which we call $g : Y \longrightarrow Y'$. By condition (ii), since $p' = g \circ p$ is continuous, so is g. Moreover, since p' is an identification and $g^{-1} \circ p' = p$ is continuous, g^{-1} is continuous. That is, g is a homeomorphism and so p is an identification.

4.2.9 Remark. As it is the case for the induced topology commented in 4.1.8, (i) is redundant. It is an *exercise* for the reader to show that (i) follows from (ii).

4.2.10 Example. If $f : I \longrightarrow \mathbb{S}^1$ is the exponential map given by $f(s) = \mathrm{e}^{2\pi \mathrm{i} s}$, then f is an identification that identifies in one point the points 0 and 1 of the interval I (see Figure 4.1).

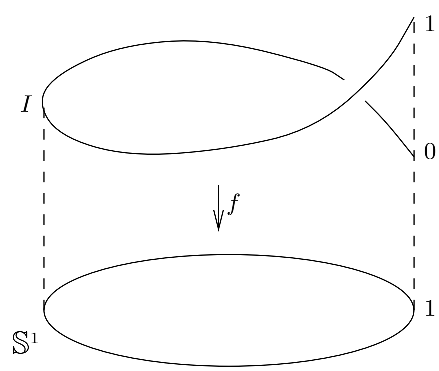


Figure 4.1 The exponential map $f : I \longrightarrow \mathbb{S}^1$

Notice that if we take $[0, 1) \subset I$ and the restriction f' of f to $[0, 1)$, then $f' : [0, 1) \longrightarrow \mathbb{S}^1$ is bijective. Thus, if it were an identification, then by 4.2.7 it would be a homeomorphism. Clearly it is not.

The property of being an identification is inherited to open or closed sets in the following sense.

4.2.11 **Theorem**. *Let $f : X \longrightarrow Y$ be an identification and take a subset Y', which is open or closed in Y and consider $X' = f^{-1}(B)$. Then*

$$f' = f|_{X'} : X' \longrightarrow Y'$$

is an identification.

Proof: Assume that $Y' \subseteq Y$ is open (resp. closed), and take $G \subseteq Y'$ such that $f'^{-1}(G) \subseteq X'$ is open (resp. closed). Since clearly $f'^{-1}(G) = f^{-1}(G)$ is open (resp. closed) in X' and X' is open (resp. closed) in X, $f^{-1}(G)$ is open (resp. closed) in X. Therefore G is open (resp. closed) in Y. Consequently, G is also open (resp. closed) in Y'. □

The following result provides a sufficient condition, used frequently, for a map to be an identification.

4.2.12 **Proposition**. *Let $p : X \longrightarrow Y$ be continuous. If there exists a continuous map $s : Y \longrightarrow X$ such that $p \circ s = \mathrm{id}_Y$, show that p is an identification. Such a map s is called a* section *of p.* □

4.2.13 NOTE.

(a) If $s : Y \longrightarrow X$ is a section of $p : X \longrightarrow Y$, then s is an embedding.

(b) Assume $Y \subseteq X$ and call the inclusion map $i : Y \hookrightarrow X$. If there is a map $r : X \longrightarrow Y$ such that $r \circ i = \mathrm{id}_Y$, then r is called a *retraction*. By (a), a retraction r is always an identification.

In what follows, we shall see an alternative way of interpreting the concept of an identification. Let X be a topological space and let $\sim$ be an equivalence relation in X. We have a surjective function onto the set of equivalence classes

$$f : X \longrightarrow X' = X/\!\sim \ .$$

4.2.14 DEFINITION. The space X' with the identification topology is called a *quotient space* of X under the relation $\sim$. One also says that X' has the *quotient topology*

4.2.15 REMARK. The quotient topology on a quotient set is exactly the dual notion to the relative topology on a subset. Similarly, the identification topology is dual to the induced topology. Correspondingly, a quotient map is the dual notion of an inclusion, and an identification is the dual notion of an embedding.

Conversely to Definition 4.2.14, if one has a surjective function $f : X \longrightarrow X'$, this function defines an equivalence relation in X given by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$. Hence, there is a bijection between $X/\sim$ with the quotient topology and X'. In fact, one has the following.

4.2.16 **Theorem**. *Let $f : X \longrightarrow X'$ be an identification. If one defines an equivalence relation in X by $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$, then f determines a homeomorphism $\widehat{f} : X/\sim \longrightarrow X'$.* □

4.2.17 EXAMPLES.

(a) If $f : \mathbb{R} \longrightarrow \mathbb{S}^1$ is again the exponential map $f(s) = \mathrm{e}^{2\pi \mathrm{i} s}$, then f is an identification such that $f(s) = f(t)$ if and only if $t - s \in \mathbb{Z}$ (see Figure 4.2).

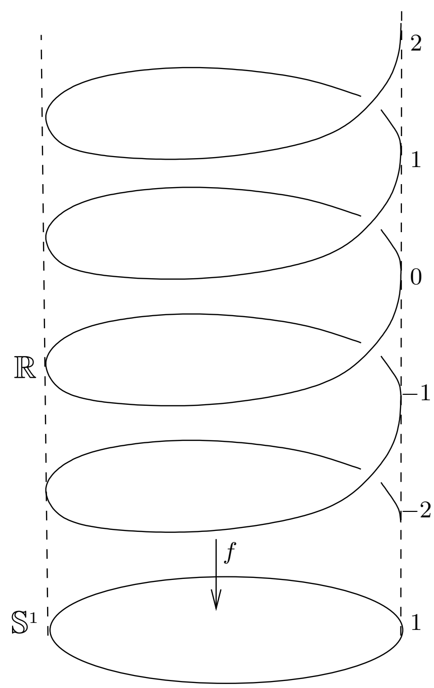


Figure 4.2 The exponential map $f : \mathbb{R} \longrightarrow \mathbb{S}^1$

(b) Let $f : I \times I \longrightarrow \mathbb{S}^1 \times I$ be the map given by $f(s,t) = (\mathrm{e}^{2\pi \mathrm{i} s}, t)$. Then f is an identification that identifies the left vertical edge of the square $I \times I$ with its right vertical edge, as shown in Figure 4.3.

(c) If we take in $\mathbb{S}^2 \subset \mathbb{R}^3$ the relation $x \sim -x$ and extend it to an equivalence relation, then the quotient space

$$\mathbb{RP}^2 = \mathbb{S}^2/\sim$$

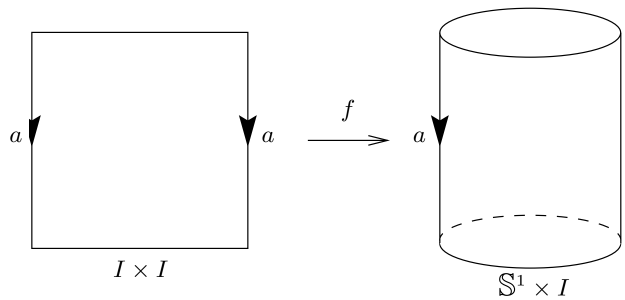


Figure 4.3 The identification $f : I \times I \longrightarrow \mathbb{S}^1 \times I$

is the space known as the *real projective plane*.

(d) More generally, if in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ we take the same relation $x \sim -x$ and extend it to an equivalence relation, then the quotient space

$$\mathbb{RP}^n = \mathbb{S}^n/\sim$$

is the space known as the *real projective space* of dimension n. In particular, the real projective space of dimension 2 is the real projective plane.

(e) There is a *complex* version of the two previous examples as follows. In $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ take the relation $x \sim x' \Leftrightarrow$ there exists $\lambda \in \mathbb{S}^1 \subset \mathbb{C}$ such that $x' = \lambda x$ (it is an *exercise* to prove that this is in fact an equivalence relation); then the quotient space

$$\mathbb{CP}^n = \mathbb{S}^{2n+1}/\sim$$

is the space known as the *complex projective space* of (complex) dimension n. In particular, the complex projective space of dimension 2 is called the *complex projective plane*.

4.2.18 NOTE. In the Examples 4.2.17 (b) and (c) above, $\sim$ means the equivalence relation *generated* by $x \sim -x$, namely the minimal equivalence relation that contains the given relation.

4.2.19 EXERCISE. Show that there is a canonical map

$$\mathbb{RP}^{2n+1} \longrightarrow \mathbb{CP}^n .$$

Prove that this map is a quotient map. (*Hint:* Use the universal property of the quotient map $\mathbb{S}^{2n+1} \longrightarrow \mathbb{RP}^{2n+1}$.)

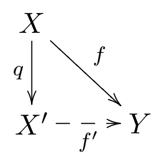
We shall see below in 4.2.25, that there are alternative ways of defining the projective spaces.

There is a dual result to 4.1.4 that characterizes the quotient topology.

4.2.20 **Proposition**. *Let X be a topological space and take $X' = X/\sim$, where $\sim$ is an equivalence relation in X. The quotient topology coinduced by X in X' is characterized by the following universal property:*

(i) *The quotient map $q : X \longrightarrow X'$ is continuous.*

(ii) *A map $f' : X' \longrightarrow Y$ is continuous if and only if the composite $f = f' \circ q$ is continuous. In a diagram*



$$f \text{ is continuous} \Leftrightarrow f' \text{ is continuous.}$$

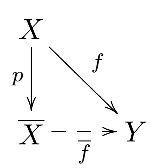
□

4.2.21 REMARK. The last result is a special case of 4.2.8 and is dual to 4.1.4.

The following theorem, which reformulates 4.2.8, gives a characterization of the identification topology in $\overline{X}$ through the *universal property of the identifications.*

4.2.22 **Theorem**. *Let $p : X \longrightarrow \overline{X}$ be continuous. Then p is an identification if and only if it has the following property:*

Assume that $f : X \longrightarrow Y$ is such that if $p(x) = p(x')$, then $f(x) = f(x')$. Then there exists a unique map $\overline{f} : \overline{X} \longrightarrow Y$ such that $\overline{f} \circ p = f$. In a diagram



$$\overline{f} \text{ is continuous} \Leftrightarrow f \text{ is continuous.}$$

Proof: Assume that p is an identification and let $f : X \longrightarrow Y$ be as in the assumption. To prove that $\overline{f}$ is continuous, let $B \subset Y$ be open. Then $\overline{f}^{-1}(B) \subset \overline{X}$ is open, since $p^{-1}\overline{f}^{-1}(B) = (\overline{f} \circ p)^{-1}(B) = f^{-1}(B)$ is open.

Conversely, take the equivalence relation $x \sim x' \Leftrightarrow p(x) = p(x')$. Let $q : X \longrightarrow X/\sim$ be the corresponding quotient map. Since q is an identification, the first part of the theorem implies that there exists a continuous map $\overline{p} : X/\sim \longrightarrow \overline{X}$ such that $\overline{p} \circ q = p$. By assumption, there exists a continuous map $\overline{q} : X/\sim \longrightarrow \overline{X}$ such that $\overline{q} \circ p = q$. It is straightforward to prove that $\overline{p} \circ \overline{q} = \mathrm{id}_{\overline{X}}$, and that $\overline{q} \circ \overline{p} = \mathrm{id}_{X/\sim}$. Hence $\overline{p}$ is a homeomorphism and since q is an identification, so is p.

□

It is possible to rewrite the statement of the previous theorem with the following concept. Given a map $p : X \longrightarrow \overline{X}$, we say that another map $f : X \longrightarrow Y$ is *compatible* with the map p if

$$p(x) = p(x') \in \overline{X} \;\Rightarrow\; f(x) = f(x') \in Y\,.$$

Then one may express the universal property of the identifications as follows.

4.2.23 **Theorem**. *$p : X \longrightarrow \overline{X}$ is an identification if and only if, given $f : X \longrightarrow Y$ compatible with p, there exists a unique $\overline{f} : \overline{X} \longrightarrow Y$ such that $\overline{f} \circ p = f$.* □

4.2.24 Definition. We say that $\overline{f}$ is the result of *passing f to the quotient.*

4.2.25 Exercise.

(a) Show that $\mathbb{RP}^n$ is homeomorphic to the quotient space of $\mathbb{R}^{n+1} - 0$ under the relation $x \sim x' \Leftrightarrow$ there exists $\lambda \in \mathbb{R}$ such that $x = \lambda x'$.

(b) Show that $\mathbb{RP}^1$ is homeomorphic to $\mathbb{S}^1$. (*Hint:* $z \mapsto z^2$ defines a map $p : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $p(z_1) = p(z_2) \Leftrightarrow z_1 = \pm z_2$.)

(c) Show that the projective plane $\mathbb{RP}^2$ is homeomorphic to the quotient space of the disk $\mathbb{D}^2$ under the equivalence relation

$$z \sim z' \leftrightarrow \begin{cases} z = z' & \text{if } |z| = |z'| < 1 \text{ or} \\ z = \pm z' & \text{if } |z| = |z'| = 1\,. \end{cases}$$

In other words, $\mathbb{RP}^2$ is obtained from $\mathbb{D}^2$ by identifying antipodal points on the boundary $\mathbb{S}^1$. (*Hint:* The disk is homeomorphic to the closed northern hemisphere $D^2_+ = \{(x, y, z) \in \mathbb{S}^2 \subset \mathbb{R}^3 \mid z \geq 0\}$.)

(d) More generally, show that the projective space $\mathbb{RP}^n$ is homeomorphic to the quotient space of the n-ball $\mathbb{D}^n$ by identifying antipodal points on the boundary $\mathbb{S}^{n-1}$.

4.2.26 Exercise.

(a) Prove that $\mathbb{CP}^n$ is homeomorphic to the quotient space of $\mathbb{C}^{n+1} - 0$ under the relation $z \sim z' \Leftrightarrow$ there exists $\lambda \in \mathbb{C}$ such that $z = \lambda z'$.

(b) Prove that $\mathbb{CP}^1$ is homeomorphic to $\mathbb{S}^2$. (*Hint:* $z = (z_1, z_2) \mapsto z_1/z_2$, $z_2 \neq 0$, defines a map $p : \{(z_1, z_2) \in \mathbb{S}^3 \mid z_2 \neq 0\} \longrightarrow \mathbb{C} \hookrightarrow \mathbb{S}^2$ such that $p(z) = p(z') \Leftrightarrow z = \lambda z'$ which can be continuously extended to a map $\mathbb{S}^3 \longrightarrow \mathbb{S}^2$. The embedding $\mathbb{C} \hookrightarrow \mathbb{S}^2$ is the inverse of the stereographic projection; see 8.3.1, below. The map $p : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ is known as the *Hopf fibration.*)

4.2.27 EXERCISE. Let X be a pseudometric space with pseudometric d and let $\widetilde{X}$ be its metric identification (see 1.5.5). Show that the topology in $\widetilde{X}$ determined by the associated metric $\widetilde{d}$ is the quotient topology corresponding to the relation on X given by $x \sim y$ if and only if $d(x,y) = 0$.

4.2.28 DEFINITION. Take $f : X \longrightarrow Y$. The map f is called an *open* (resp. *closed*) map if for every open (resp. closed) subset $A \subset X$, the image set $f(A)$ is open (resp. closed) in Y.

The following is a very useful result to determine when a given map is open; the proof is obvious.

4.2.29 **Theorem**. *Let $f : X \longrightarrow Y$ be continuous. If $f(A)$ is open for every basic open set A of X, then f is open.* □

4.2.30 EXAMPLES.

(a) $f : X \longrightarrow Y$ is a homeomorphism if and only if f is continuous, bijective, and open (resp. closed).

(b) The projection $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ onto some k of its n coordinates is open; however, it is not closed. For example, the first projection $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is not closed, because taking $A = \{(x,y) \in \mathbb{R}^2 \mid y = \frac{1}{x},\ x > 0\}$, A is closed in $\mathbb{R}^2$; however, its image $\pi(A) = (0, +\infty)$, as shown in Figure 4.4, is not closed in $\mathbb{R}$.

(c) The function $\sin : [0, 2\pi] \longrightarrow [-1, 1]$ is a closed map but it is not open, because if we take $A = [0, \frac{3\pi}{4})$ that is open in $[0, 2\pi]$, its image $f(A) = [0, 1]$ is not open in $[-1, 1]$.

(d) Let $f : X \longrightarrow Y$ be surjective such that X has the topology induced through f, then f is an open and closed map.

Examples (b) and (c) above show that the concepts of an open and a closed map are independent.

4.2.31 EXERCISE. Let $f : X \longrightarrow Y$ be a map between topological spaces. Prove that the following are equivalent:

(a) f is closed.

(b) For every set A open in X, the set $\{y \in Y \mid f^{-1}(y) \subset A\}$ is open in Y.

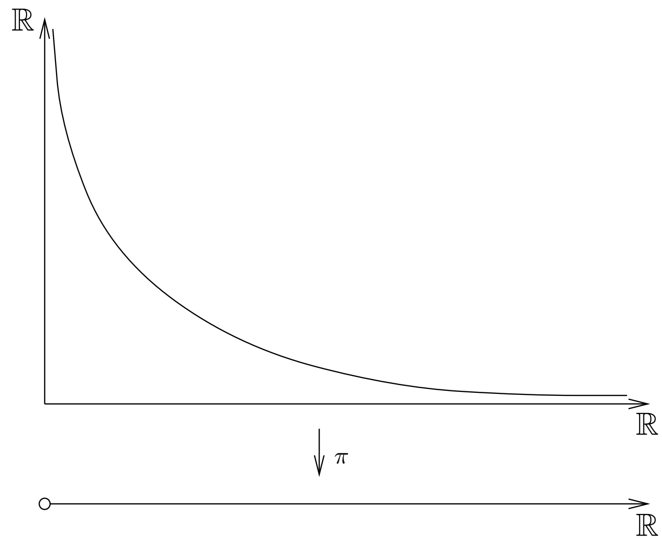


Figure 4.4 The projection $\pi : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ onto the first factor maps the hyperbolic branch onto an open set.

(c) For every set C closed in X, the set $\{y \in Y \mid f^{-1}(y) \cap C \neq \emptyset\}$ is closed in Y.

(d) For every set B in X, $\overline{f(B)} \subset f(\overline{B})$; indeed, $\overline{f(B)} = f(\overline{B})$.

4.2.32 Exercise. Let $f : X \longrightarrow Y$ be a map between topological spaces. Prove that the following are equivalent:

(a) f is open.

(b) For every set B in X, $f(B^\circ) \subset f(B)^\circ$.

4.2.33 Exercise. Prove that a map $f : X \longrightarrow Y$ is open (resp. closed) if and only if for every subset $T \subset Y$ the restriction $f_T = f|_{f^{-1}(T)} : f^{-1}(T) \longrightarrow T$ is open (resp. closed).

4.2.34 **Theorem**. *If $f : X \longrightarrow Y$ is continuous, surjective, and open (resp. closed), then f is an identification.*

Proof: Since f is surjective, for every $B \subset Y$, $f(f^{-1}(B)) = B$. Hence, let B be such that $f^{-1}(B)$ is open (resp. closed), then $B = f(f^{-1}(B))$ is open (resp. closed). Therefore, f is an identification. □

As we saw in the proof of Theorem 4.2.22, every identification is, up to homeomorphism, a quotient map.

4.2.35 DEFINITION. Let $p : X \longrightarrow X/\sim$ be a quotient map (identification). If $A \subset X$, we define its *saturation* with respect to $\sim$ as $p^{-1}(p(A))$. This set consists of all points of A together with all those points of X that are equivalent to some point of A. If A is such that $A = p^{-1}(p(A))$, then we say that A is *saturated* with respect to $\sim$.

4.2.36 **Proposition**. *Let $A \subset X$ be open (resp. closed) and saturated with respect to a relation $\sim$ and let $p : X \longrightarrow X/\sim$ be the coresponding quotient map. Then $p|_A : A \longrightarrow p(A)$ is an identification.*

Proof: Let $B \subset p(A)$ be such that $(p|_A)^{-1}(B) = p^{-1}(B)$ is open (resp. closed) in A, and therefore, also in X. Since p is an identification, B is open (resp. closed) in $X/\sim$ and hence also in $p(A)$. □

4.2.37 EXERCISE. Let $f : X \longrightarrow Y$ be continuous.

(a) Prove that f factorizes as

$$X \xrightarrow{f^*} Z^* \overset{e}{\hookrightarrow} Y\,,$$

where f^* is surjective and e is an embedding.

(b) Prove that f factorizes as

$$X \overset{q}{\twoheadrightarrow} Z_* \xrightarrow{f_*} Y\,,$$

where f_* is injective and q is a quotient map.

To finish this section, we shall consider a particular case of quotient maps that are frequently found in topology. First recall that to give an equivalence relation on a set X is equivalent to give a *partition* of X, namely a family of disjoint subsets of X whose union is X. The correspondence is given by taking the equivalence classes as subsets of the partition. We shall consider a particular case of quotient maps that are frequently found in topology.

4.2.38 DEFINITION. Let X be a topological space and $A \subset X$. Consider the partition consisting of the set A and all singular sets $\{x\}$ for all $x \notin A$ (this is equivalent to the equivalence relation given by $x \sim x'$ if and only if either $x, x' \in A$ or $x = x'$. We denote the quotient space $X/\sim$ by X/A and we say that X/A is obtained from X by *collapsing* A into a one-point space that we frequently denote either by $\{A\}$ or simply by $*$.

4.2.39 EXAMPLES.

1. Consider the unit disk $\mathbb{D}^2 \subset \mathbb{C}$ and define the following partition:

   (i) If $|z| < 1$, then $\{z\}$ belongs to the partition.

   (ii) If $|z| = 1$, then $z = \mathrm{e}^{2\pi \mathrm{i}t}$ for some $t \in I$. Then consider the two-element sets

   $$\{\mathrm{e}^{2\pi\mathrm{i}(4k+t)/4n}, \mathrm{e}^{2\pi\mathrm{i}(4k+3-t)/4n}\}, \quad \{\mathrm{e}^{2\pi\mathrm{i}(4k+1+t)/4n}, \mathrm{e}^{2\pi\mathrm{i}(4k+4-t)/4n}\},$$

   where $t \in I$ and $k = 0, \ldots, n-1$, as the other sets of the partition.

The resulting quotient space is the *closed orientable surface of genus* n. Alternatively, we may describe this construction as follows. Take the points in $\mathbb{S}^1$ defined by

$$p_k = \mathrm{e}^{2\pi\mathrm{i}k/4n}, \text{ where } k = 0, 1, 2, \ldots, 4n-1,$$

that is, the $4n$th roots of unit. Then the arcs on the the boundary of the disk are identified as shown in Figure 4.5 for the case $n = 2$. This means that the first arc labeled a_1 from p_0 to p_1 is identified with the second arc labeled $\overline{a}_1$ which is oriented in the opposite direction from p_3 to p_2. Then the arc labeled b_1 from p_1 to p_2 is identified with the second arc labeled $\overline{b}_2$ in the opposite direction from p_4 to p_3 and so on.

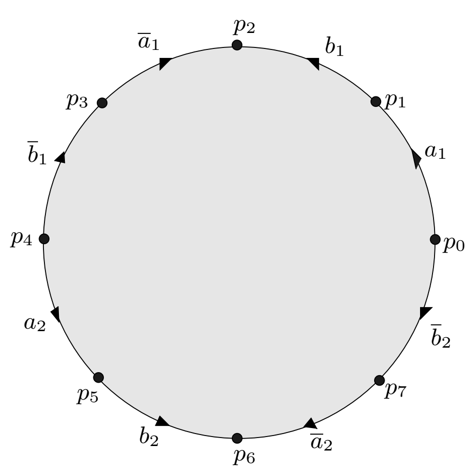


Figure 4.5 Octagon

In the case $n = 1$ this example provides the construction of the torus. See 1.5.7(b) above or 5.4.12(d) below. Figure 4.6 shows the construction for $n = 1$ and $n = 2$ using polygons instead of the disk. The edges correspond to the arcs up to a homeomorphism.

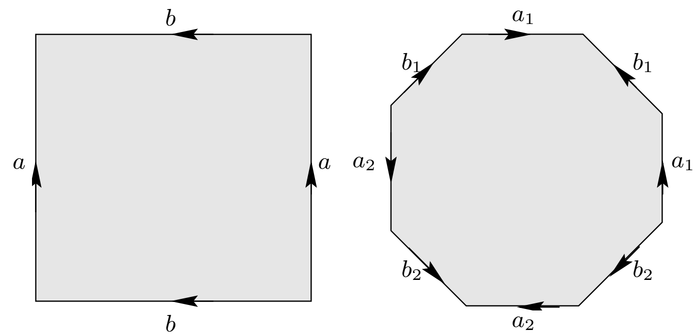


Figure 4.6 Polygonal cases $n = 1$ and $n = 2$

2. Consider the unit disk $\mathbb{D}^2 \subset \mathbb{C}$ again and now define define the following partition:

   (i) If $|z| < 1$, then $\{z\}$ belongs to the partition.

   (ii) If $|z| = 1$, then $z = \mathrm{e}^{2\pi\mathrm{i}t}$ for some $t \in I$. Then consider the two-element sets
$$\{\mathrm{e}^{2\pi\mathrm{i}(2k+t)/2n}, \mathrm{e}^{2\pi\mathrm{i}(2k+1+t)/2n}\},$$
   where $t \in I$ and $k = 0, \ldots, n-1$, as the other sets of the partition.

   The resulting quotient space is the *closed nonorientable surface of genus n*. Alternatively, we may describe this construction as follows. Take the points in $\mathbb{S}^1$ defined by
$$p_k = \mathrm{e}^{2\pi\mathrm{i}k/2n}, \text{ where } k = 0, 1, 2, \ldots, 2n-1,$$
   that is, the $2n$th roots of unit. Then the arcs on the the boundary of the disk are identified as shown in Figure 4.7 for the case $n = 3$. This means that the first arc labeled a_1 from p_0 to p_1 is identified with the second arc labeled a_1' from p_1 to p_2. Then the arc labeled a_2 from p_2 to p_3 is identified with arc labeled a_2' from p_3 to p_4 and so on.

   In the case $n = 1$ this example provides the construction of the projective plane given in 4.2.25 (c). Figure 4.8 shows the construction for $n = 2$ (the Klein bottle, see 5.4.20) and $n = 3$ using polygons instead of the disk. As in the previous case, the edges correspond to the arcs up to a homeomorphism.

4.2.40 Exercise. Show that $I/\{0,1\} \approx \mathbb{S}^1$. (*Hint:* See Example 4.2.17(a).)

More generally, we have the following.

4.2.41 Exercise. Show that there is a homeomorphism $\mathbb{B}^n/\mathbb{S}^{n-1} \approx \mathbb{S}^n$.

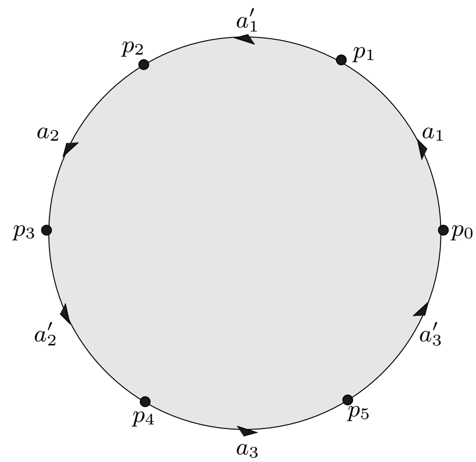


Figure 4.7 Hexagon

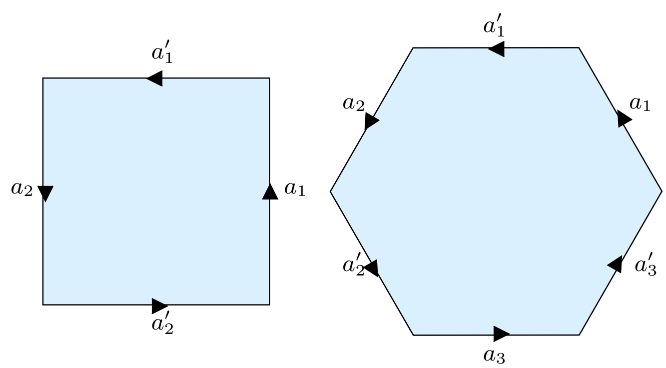


Figure 4.8 Polygonal cases $n = 2$ and $n = 3$

Convention. If $A = \emptyset$ we convene to define $X/\emptyset$ as the space $X^+ = X \sqcup \{*\}$, namely the space X together with an isolated point $*$. See Section 4.4 for the general usage of the symbol $\sqcup$.

To finish this section it is convenient to remark the duality between certain notions related to the induced topology and those related to the coinduced topology:

(a) *Relative topology* is dual to *quotient topology*. Thus *inclusion* is dual to *quotient map*.

(b) *Embedding* is dual to *identification*.

(c) *Restriction* is dual to *pass to the quotient* (see 4.2.24).

4.3 Topological product

Let us start by recalling what the cartesian product of an arbitrary family of sets is. Given a collection $\{X_\lambda \mid \lambda \in \Lambda\}$ of sets, we define their *cartesian product* by

$$\prod_{\lambda\in\Lambda} X_\lambda = \{x : \Lambda \longrightarrow \bigcup X_\lambda \mid x(\lambda) \in X_\lambda\}\,.$$

Given $\mu \in \Lambda$, we define the *projection* $p_\mu : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\mu$ by $p_\mu(x) = x(\mu)$. It is usual to denote $x(\lambda)$ by x_λ and x by $(x_\lambda)_{\lambda\in\Lambda}$ or simply by (x_λ).

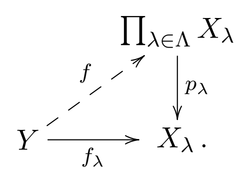
If Λ is finite, say $\Lambda = \{1, \dots, k\}$ one usually denotes the cartesian product by

$$X_1 \times \cdots \times X_k\,.$$

The cartesian product has the following universal property.

4.3.1 **Theorem**. *Given a family $\{X_\lambda \mid \lambda \in \Lambda\}$ of sets, the cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$ together with the projections $p_\mu : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\mu$, $\mu \in \Lambda$, is characterized by the following property:*

(CP) *Given any functions $f_\lambda : Y \longrightarrow X_\lambda$, $\lambda \in \Lambda$, there is a unique function $f : Y \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ such that $p_\lambda \circ f = f_\lambda$. In a diagram*



Proof: Given $y \in Y$, define $f(y) \in \prod_{\lambda\in\Lambda} X_\lambda$ by $f(y)(\lambda) = f_\lambda(y)$. In other words, $f(y) = (f_\lambda(y))_{\lambda\in\Lambda}$.

If there are a set X and functions $q_\lambda : X \longrightarrow X_\lambda$ which fulfill (CP), namely such that given any functions $f_\lambda : Y \longrightarrow X_\lambda$, there is a function $f : Y \longrightarrow X$ such that $q_\lambda \circ f = f_\lambda$, then there is a bijective function $\varphi : X \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ such that $p_\lambda \circ \varphi = q_\lambda$. This can be easily verified using (CP) for both families of functions p_λ and q_λ. We leave the details as an *exercise* to the reader. □

We now pass to the topological case. Thus we assume now that we have a family $\{X_\lambda \mid \lambda \in \Lambda\}$ of topological spaces. We wish to furnish $\prod_{\lambda\in\Lambda} X_\lambda$ with a topology such that the following two conditions hold:

(i) The projection $p_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\lambda$ is continuous for all $\lambda \in \Lambda$.

(ii) Given a continuous map $f_\lambda : Y \longrightarrow X_\lambda$ for each $\lambda \in \Lambda$, the function $f : Y \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ such that $p_\lambda \circ f = f_\lambda$ is also continuous.

First we deal with condition (i) considering the following question: Given a set X and for each $\lambda \in \Lambda$ a function $p_\lambda : X \longrightarrow X_\lambda$, which is the coarsest topology in X that makes every p_λ continuous? As we already saw above, this topology is the least upper bound, i.e., the supremum of all topologies induced in X by each of the functions p_λ (compare this with Exercise 4.1.30(a)). A subbasis for this topology consists of all sets of the form $p_\lambda^{-1}(Q_\lambda)$, where $Q_\lambda \subseteq X_\lambda$ is open, $\lambda \in \Lambda$. We are interested in the following special case. Let $X = \prod_{\lambda\in\Lambda} X_\lambda$, be the cartesian product, seen as a set, of a family $\{X_\lambda \mid \lambda \in \Lambda\}$ of topological spaces. The general question stated above gives origin in this case to the problem that we want to solve. We want to construct the coarsest topology on X for which all projections become continuous. In the case of the product we have $p_\lambda^{-1}(Q_\lambda) = Q_\lambda \times \prod_{\mu\neq\lambda} X_\mu$. As we already know, the supremum of a family of topologies has as a basis the finite intersections of subbasic sets. Hence a finite intersection of subbasic open sets is a product of the form

$$Q_{\lambda_1} \times \cdots \times Q_{\lambda_k} \times \prod_{\mu\neq\lambda_1,\ldots\lambda_k} X_\mu \,, \tag{4.3.2}$$

where Q_{λ_i} is open in X_{λ_i}, $i = 1, \ldots, k$, (see Figure 4.9). Sometimes we denote the basic sets $Q_{\lambda_1} \times \cdots \times Q_{\lambda_k} \times \prod_{\mu\neq\lambda_1,\ldots\lambda_k} X_\mu$ by $\prod_{\lambda\in\Lambda} Q_\lambda$, where $Q_\lambda = X_\lambda$ for almost every $\lambda \in \Lambda$. We have the following.

4.3.3 Definition. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a family of topological spaces and consider their cartesian product $X = \prod_{\lambda\in\Lambda} X_\lambda$. The *product topology* on X has as a basis the family of sets

$$Q_{\lambda_1} \times \cdots Q_{\lambda_k} \times \prod_{\mu\neq\lambda_1,\ldots\lambda_k} X_\mu \,.$$

The cartesian product with this topology is called *topological product* of the spaces X_λ, and it is usually denoted by $\prod_{\lambda\in\Lambda} X_\lambda$.

By construction, condition (i) holds. Namely, we have the following.

4.3.4 **Theorem**. *Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a family of topological spaces. If $\prod_{\lambda\in\Lambda} X_\lambda$ has the product topology, then all projections $p_\mu : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\mu$, $\mu \in \Lambda$, are continuous.*

The *proof* is immediate by noting that a basic set has the form (4.3.2). $\square$

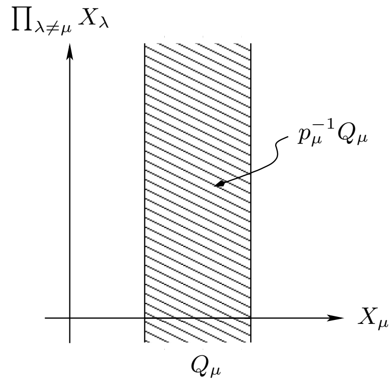


Figure 4.9 A subbasic set in the product

4.3.5 Definition. The basis for the product topology on $\prod_{\lambda \in \Lambda} X_\lambda$ given by all sets of the form (4.3.2) is called *natural basis* for the topological product.

4.3.6 Note. In case that the index set Λ is finite, say $\Lambda = \{1, 2, \ldots, m\}$, the natural basis for the product topology in $\prod_{\lambda \in \Lambda} X_\lambda$ consists of the products $\prod_{\lambda \in \Lambda} Q_\lambda$, where Q_λ is open in X_λ, $\lambda = 1, \ldots, m$. In this case, one usually denotes the topological product by

$$X_1 \times X_2 \times \cdots \times X_m$$

and its natural basic open sets by $Q_1 \times \cdots \times Q_m$. We call these products *boxes.*

4.3.7 Note. In case that the index set Λ is infinite, we call the canonical basic sets

$$Q_{\lambda_1} \times \cdots \times Q_{\lambda_k} \times \prod_{\mu \neq \lambda_1, \ldots \lambda_k} X_\mu$$

finite boxes. One may define a different topology on the product if we take *infinite boxes* of the form $\prod_{\lambda \in \Lambda} Q_\lambda$, where $Q_\lambda \subseteq X_\lambda$ is an open set. This yields a finer topology on the cartesian product, and so the projections are continuous. But this topology will in general not fulfill condition (ii). This topology is called the *box topology*

We now prove that the product topology satisfies condition (ii).

4.3.8 **Proposition**. *Let X_λ, $\lambda \in \Lambda$, be a family of topological spaces and let $f_\lambda : Y \longrightarrow X_\lambda$ be a family of continuous maps. Then the unique map $f : Y \longrightarrow \prod_{\lambda \in \Lambda} X_\lambda$ such that $p_\lambda \circ f = f_\lambda$, is continuous.*

Proof: It is enough to take a basic open set $Q_{\lambda_1} \times \ldots Q_{\lambda_k} \times \prod_{\lambda \neq \Lambda_i} X_\lambda$ and prove that its inverse image is open. Since

$$f^{-1}(Q_{\lambda_1} \times \ldots Q_{\lambda_k} \times \prod_{\lambda \neq \Lambda_i} X_\lambda) = f_{\lambda_1}^{-1}(Q_{\lambda_1}) \cap \cdots \cap f_{\lambda_k}^{-1}(Q_{\lambda_k}) \,,$$

and each f_λ is continuous, this inverse image is a finite intersection of open sets, thus it is open. □

4.3.9 Note. If we take the box topology in the product, then we must take infinite intersections of open sets, which in general will not be open. Thus f need not be continuous. It is in this sense that the product topology is the right one in order to have the universal property, which we show below.

4.3.10 Example. $\mathbb{R}^n$ is the topological product of n copies of the topological space $\mathbb{R}$.

4.3.11 Note. If $X_\lambda = Y$, $\lambda \in \Lambda$, then one usually denotes the product $\prod_{\lambda\in\Lambda} X_\lambda$ simply by Y^Λ.

4.3.12 Example. Let $\mathbb{R}_k = \mathbb{R}$, $k \in \mathbb{N}$, and take $f_k = \mathrm{id}_\mathbb{R} : \mathbb{R} \longrightarrow \mathbb{R}_k$, $k \in \mathbb{N}$. By the universal property of the product, there is a continuous map $f : \mathbb{R} \longrightarrow \prod_{k\in\mathbb{N}} \mathbb{R}_k$ such that $p_k \circ f = f_k = \mathrm{id}_\mathbb{R}$. Now consider the set $\prod_k(-\frac{1}{k}, \frac{1}{k}) \subset \prod_k \mathbb{R}_k$. Its inverse image under f is

$$f^{-1}(\prod_k(-\frac{1}{k}, \frac{1}{k})) = \bigcap_k(-\frac{1}{k}, \frac{1}{k}) = \{0\}\,,$$

which is not an open set. Since f is continuous, the product $\prod_k(-\frac{1}{k}, \frac{1}{k}) \subset \prod_k \mathbb{R}_k$ is not an open set. This shows that the box topology on the product $\prod_k \mathbb{R}_k$ is strictly finer than the product topology.

4.3.13 Exercise. Prove that if Λ is not countable and X_λ is not indiscrete for all λ, then $\prod_{\lambda\in\Lambda} X_\lambda$ does not satisfy any of the two countability axioms. However, if Λ is countable and X_λ satisfies the first, resp. the second, countability axiom, prove that $\prod_{\lambda\in\Lambda} X_\lambda$ satisfies the first, resp. the second, countability axiom.

Take $\lambda \in \Lambda$, $p_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\lambda$, and a basic open set $\prod_{\lambda\in\Lambda} Q_\lambda$. Then $p_\lambda(\prod_{\lambda\in\Lambda} Q_\lambda) = Q_\lambda$ is open. Hence, by 4.2.29 and 4.3.8, we have the next theorem.

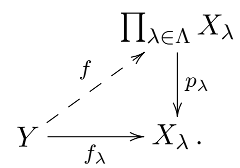
4.3.14 **Theorem**. *Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a family of topological spaces. The projections $p_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\lambda$, $\lambda \in \Lambda$, are continuous and open maps.* □

4.3.15 Note. If $X_\lambda \neq \emptyset$ for every $\lambda \in \Lambda$, then $p_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\lambda$ is surjective. Therefore Theorem 4.3.14, together with 4.2.34, implies that p_λ is an identification. (In fact, p_λ identifies all points x in $\prod_{\lambda\in\Lambda} X_\lambda$ that have the same coordinate x_λ.)

The topological product together with its projections has the following universal property.

4.3.16 **Theorem**. *The topological product* $\prod_{\lambda\in\Lambda} X_\lambda$ *together with its projections* $p_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\lambda$ *is characterized by the fact that the projections* p_λ *are continuous for all* λ *and the following property.*

(TP) *For every family* $\{f_\lambda : Y \longrightarrow X_\lambda \mid \lambda \in \Lambda\}$ *of continuous maps, there exists a unique continuous map* $f : Y \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ *such that* $p_\lambda \circ f = f_\lambda$. *In a diagram*



$$f \text{ is continuous} \Leftrightarrow f_\lambda \text{ is continuous } \forall\lambda$$

Proof: It is enough to prove that if a topological space X, together with a family of continuous maps $q_\lambda : X \longrightarrow X_\lambda$, satisfies (TP), that is, it is such that given any family $\{f_\lambda : Y \longrightarrow X_\lambda \mid \lambda \in \Lambda\}$ of continuous maps, there exists a unique continuous map $f : Y \longrightarrow X$ such that $q_\lambda \circ f = f_\lambda$, then there must exist a homeomorphism between $\prod_{\lambda\in\Lambda} X_\lambda$ and X that commutes with the projections.

By the universal property of $\prod_{\lambda\in\Lambda} X_\lambda$, there exists a unique $q : X \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ such that $p_\lambda \circ q = q_\lambda$. Analogously, by the universal property for X, there exists a unique $p : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X$ such that $q_\lambda \circ p = p_\lambda$. It is an immediate consequence of the uniqueness demanded by the universal property of both X and $\prod_{\lambda\in\Lambda} X_\lambda$ that the composites $p \circ q$ and $q \circ p$ are the respective identity maps, that is, p and q are inverse homeomorphisms that commute with the projections. □

Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a family of topological spaces and $A_\lambda \subseteq X_\lambda$, $\lambda \in \Lambda$ be subspaces. If we take $\prod_{\lambda\in\Lambda} A_\lambda \subset \prod_{\lambda\in\Lambda} X_\lambda$, then the cartesian product $\prod_{\lambda\in\Lambda} A_\lambda$ has two ways of getting a topology. One consists in furnishing each A_λ with the relative topology and then giving $\prod_{\lambda\in\Lambda} A_\lambda$ the product topology. The other consists in furnishing $\prod_{\lambda\in\Lambda} A_\lambda$ with the relative topology induced by that of $\prod_{\lambda\in\Lambda} X_\lambda$. We have the following.

4.3.17 **Proposition**. *Both the relative and the product topology on* $\prod_{\lambda\in\Lambda} A_\lambda$ *are equal.*

Proof: A basic set in the relative topology has the form $\prod_{\lambda\in\Lambda} A_\lambda \cap \prod_{\lambda\in\Lambda} Q_\lambda$. But such set clearly coincides with $\prod(A_\lambda \cap Q_\lambda)$, which is a natural basic set in the product topology $\prod_{\lambda\in\Lambda} A_\lambda$. □

4.3.18 Exercise. As an alternative proof of the previous proposition, show that $\prod_\lambda A_\lambda \subseteq \prod_\lambda X_\lambda$ with the relative topology has the universal property 4.3.16.

If $Q_\lambda \subset X_\lambda$ is open for all λ, then in general $\prod_{\lambda\in\Lambda} Q_\lambda$ is not open, as we saw in Example 4.3.12. However we have the following.

4.3.19 **Lemma**. *Let $A_\lambda \subseteq X_\lambda$ be a closed set for all $\lambda \in \Lambda$. Then $\prod_{\lambda\in\Lambda} A_\lambda \subseteq \prod_{\lambda\in\Lambda} X_\lambda$ is closed.*

Proof: Consider the complement

$$\prod_{\lambda\in\Lambda} X_\lambda - \prod_{\lambda\in\Lambda} A_\lambda = \bigcup_\lambda \left[(X_\lambda - A_\lambda) \times \prod_{\lambda'\neq\lambda} X_{\lambda'} \right] .$$

Since $A_\lambda \subset X_\lambda$ is closed for every $\lambda \in \Lambda$, this complement is an open set in $\prod_{\lambda\in\Lambda} X_\lambda$. Thus $\prod_{\lambda\in\Lambda} A_\lambda$ is closed.

As a consequence of this we have the next.

4.3.20 **Proposition**. *If $A_\lambda \subseteq X_\lambda$, $\lambda \in \Lambda$, then*

$$\overline{\prod A_\lambda} = \prod \overline{A_\lambda} .$$

Proof: Assume that $\prod A_\lambda \neq \emptyset$. Since $A_\lambda \subseteq \overline{A_\lambda}$, one has $\prod A_\lambda \subseteq \prod \overline{A_\lambda}$. But the term on the right is closed by the previous lemma. Hence $\overline{\prod A_\lambda} \subseteq \prod \overline{A_\lambda}$.

Conversely, if $x \in \prod \overline{A_\lambda}$, then $x_\lambda \in \overline{A_\lambda}$ for all λ. Take a basic open neighborhood of x of the form $\prod Q_\lambda$, where $Q_\lambda = X_\lambda$ for almost all λ. Then Q_λ is an open neighborhood of x_λ for every λ. Hence $Q_\lambda \cap A_\lambda \neq \emptyset$. Therefore

$$\prod Q_\lambda \cap \prod A_\lambda = \prod (Q_\lambda \cap A_\lambda) \neq \emptyset ,$$

and so $x \in \overline{\prod A_\lambda}$. Hence $\prod \overline{A_\lambda} \subseteq \overline{\prod A_\lambda}$.

By the previous results we obtain the following.

4.3.21 **Theorem**. *Take $A_\lambda \subset X_\lambda$ such that $A_\lambda \neq \emptyset$ for every λ. Then $\prod_{\lambda\in\Lambda} A_\lambda$ is closed in $\prod_{\lambda\in\Lambda} X_\lambda$ if and only if A_λ is closed in X_λ for every $\lambda \in \Lambda$.*

Proof: If $A_\lambda \subseteq X_\lambda$ is closed, then by Lemma 4.3.19, $\prod_{\lambda\in\Lambda} A_\lambda$ is closed in $\prod_{\lambda\in\Lambda} X_\lambda$.

Conversely, if $\prod_{\lambda\in\Lambda} A_\lambda \subseteq \prod_{\lambda\in\Lambda} X_\lambda$ is closed, then by Proposition 4.3.20, $\prod_{\lambda\in\Lambda} A_\lambda = \overline{\prod_{\lambda\in\Lambda} A_\lambda} = \prod \overline{A_\lambda}$. Since $A_\lambda \neq \emptyset$ for all λ, we have

$$A_\lambda = p_\lambda(\prod A_\lambda) = p_\lambda(\prod \overline{A_\lambda}) = \overline{A_\lambda} .$$

Hence $A_\lambda \subseteq X_\lambda$ is closed for all λ.

Given the product of two topological spaces $X \times Y$, with $X \neq \emptyset$, we can view Y as a subspace of $X \times Y$ through the embedding $i_{x_0} : Y \longrightarrow X \times Y$ given by $i_{x_0}(y) = (x_0, y)$. If X is a T_1 space, that is, if every point in X is closed (see 7.1.37), then these subspaces are closed and their union is $X \times Y$, in other words, they constitute a *closed cover* of $X \times Y$.

4.3.22 Exercise. Prove that, in fact, i_{x_0} is an embedding. (*Hint:* The projection p_Y restricted to the image $i_{x_0}(Y) = \{x_0\} \times Y$ is the inverse of i_{x_0}.)

4.3.23 Exercise. More generally, prove that if $\{X_\lambda \mid \lambda \in \Lambda\}$ is a familiy of nonempty topological spaces and $x_\lambda^0 \in X_\lambda$ is a given point for every $\lambda \neq \kappa$, then the map $i_\kappa : X_\kappa \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ such that $i_\kappa(x_\kappa) = (x_\lambda)_{\lambda\in\Lambda}$, where $x_\lambda = x_\lambda^0$ if $\lambda \neq \kappa$, is an embedding.

Take $f : X \times Y \longrightarrow Z$. If f is continuous and $x_0 \in X$, then the function $f_{x_0} = f \circ i_{x_0} : Y \longrightarrow Z$ given by $f_{x_0}(y) = f(x_0, y)$ is continuous. However, the converse is false.

4.3.24 Exercise. Take $X = Y = Z = \mathbb{R}$ and define

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0). \end{cases}$$

(a) Show that if x_0 is a given point, then f_{x_0} is continuous.

(b) Compute the restriction of f to the straight line $x = y$, namely the composite $f \circ \Delta : \mathbb{R} \longrightarrow \mathbb{R}$, where $\Delta(x) = (x, x)$.

(c) Show that f is not continuous.

Let X_λ and Y_λ, $\lambda \in \Lambda$, be two nonempty families of nonempty topological spaces and take $f_\lambda : X_\lambda \longrightarrow Y_\lambda$. Consider the composites

$$\prod_{\lambda\in\Lambda} X_\lambda \xrightarrow{p_\lambda} X_\lambda \xrightarrow{f_\lambda} Y_\lambda \,,$$

where p_λ is the projection. By the universal property of the product $\prod_{\lambda\in\Lambda} Y_\lambda$, there exists a unique map $\prod f_\lambda : \prod X_\lambda \longrightarrow \prod Y_\lambda$, that makes the next diagram commutative

$$\begin{array}{ccc} \prod X_\lambda & \xrightarrow{\prod f_\lambda} & \prod Y_\lambda \\ {\scriptstyle p_\lambda}\downarrow & & \downarrow{\scriptstyle q_\lambda} \\ X_\lambda & \xrightarrow[f_\lambda]{} & Y_\lambda \,, \end{array}$$

where q_λ is the projection. Indeed, $\prod f_\lambda$ is given by $(\prod f_\lambda)((x_\lambda)_{\lambda\in\Lambda}) = (f_\lambda(x_\lambda))_{\lambda\in\Lambda}$.

4.3.25 **Proposition**. *$\prod f_\lambda$ is continuous if and only if f_λ is continuous for every $\lambda \in \Lambda$.*

Proof: Since $q_\lambda \circ \prod f_\lambda = f_\lambda \circ p_\lambda$, (TP) implies that $\prod f_\lambda$ is continuous if f_λ is continuous. Conversely, $f_\lambda = q_\lambda \circ \prod f_\lambda \circ i_\lambda$, where $i_\lambda : X_\lambda \longrightarrow \prod X_\lambda$ is such that $i_\lambda(x_\lambda) = (x_\mu)_{\mu\in\Lambda}$, with x_μ given for every $\mu \neq \lambda$, that is clearly continuous (see Exercise 4.3.23). Therefore, if $\prod f_\lambda$ is continuous, then f_λ is also continuous. □

4.3.26 Exercise. Let X be a discrete topological space and let Λ be an infinite set. Analyze the topology of the product X^Λ of as many copies of X as elements $\lambda \in \Lambda$. Is this a discrete space? If not in general, say when; otherwise, describe its topology.

Consider the *inverse system* of maps of topological spaces

$$\cdots \longrightarrow X_{n+1} \xrightarrow{h_n^{n+1}} X_n \longrightarrow \cdots \longrightarrow X_3 \xrightarrow{h_2^3} X_2 \xrightarrow{h_1^2} X_1 \tag{4.3.26}$$

and take the following subspace of the product:

$$X = \{(x_n) \mid h_n^{n+1}(x_{n+1}) = x_n\} \subset \prod_n X_n\,.$$

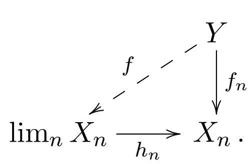
4.3.27 Definition. The space X defined above is called the *limit* of the inverse system (4.3.26) and is denoted by $\lim_n X_n$.* This space is equipped with maps $h_n : \lim_n X_n \longrightarrow X_n$ defined as the restrictions of the canonical projections

$$h_n : \lim_n X_n \xhookrightarrow{\;i\;} \prod_n X_n \xrightarrow{\;p_n\;} X_n\,,$$

that is, $h_n = p_n|_X$. These maps satisfy $h_m^n \circ h_n = h_m : \lim_n X_n \longrightarrow X_m$, where $h_m^n = h_{n-1}^n \circ \cdots \circ h_m^{m+1}$, $m < n$. The topology of $\lim_n X_n$ is often referred to as the *limit topology.*

The limit has the following universal property, that characterizes it.

4.3.28 **Theorem**. *Let $\{f_n : Y \longrightarrow X_n \mid n \geq 1\}$ be a family of maps such that $h_n^{n+1} \circ f_{n+1} = f_n : Y \longrightarrow X_n$ for every $n \geq 1$, or equivalently, $h_m^n \circ f_n = f_m : Y \longrightarrow X_m$ for every $n > m \geq 1$. Then there exists a unique map $f : Y \longrightarrow \lim_n X_n$ such that $h_n \circ f = f_n$; in a diagram*



*Some authors call this the *inverse limit* and denote it by $\varprojlim X_n$ or inv lim X_n.

Proof: By the universal property of the topological product 4.3.16, the maps f_n determine a unique map $f' : Y \longrightarrow \prod_n X_n$ such that $p_n \circ f' = f_n$. Since $h_n^{n+1} \circ f_{n+1} = f_n$, $h_n^{n+1}(f_{n+1}(y)) = f_n(y)$; therefore, $f'(y) = (f_n(y))$ lies in $\lim_n X_n$, and thus, since this space has the relative topology, f' determines a map $f : Y \longrightarrow \lim_n X_n$ with the desired properties. □

4.3.29 Exercise. Prove that the property of the space $\lim_n X_n$ just given in 4.3.28 characterizes the limit.

4.3.30 Exercise. Prove that given an inverse system of inclusions of subspaces

$$\cdots \subset X_{n+1} \subset X_n \subset \cdots \subset X_3 \subset X_2 \subset X_1 ,$$

one has that $\lim_n X_n = \bigcap_n X_n$, with the relative topology with respect to any of the inclusions $\bigcap_n X_n \subset X_n$.

4.3.31 Exercise. Prove that given an inverse system

$$\cdots \longrightarrow X_{n+1} \xrightarrow{h_n^{n+1}} X_n \longrightarrow \cdots \longrightarrow X_3 \xrightarrow{h_2^3} X_2 \xrightarrow{h_1^2} X_1 ,$$

one has that the limit topology of $\lim_n X_n$ is the coarsest that makes all the maps $h_n : \lim_n X_n \longrightarrow X_n$ continuous. (*Hint:* If $X = \lim_n X_n$ has a coarser topology that makes all the maps h_n continuous, then by 4.3.28, $\mathrm{id} : X \longrightarrow \lim_n X_n$ is continuous, and hence the topology of X is also finer than the limit topology of $\lim_n X_n$.)

Below, in Chapter 5, we shall see a more general concept of a limit of a diagram of topological spaces.

4.4 Topological sum

In this section we want to study the topological sum of a family of topological spaces. This concept is dual to the topological product. Thus we start defining the *disjoint union* of a family $\{X_\lambda \mid \lambda \in \Lambda\}$ sets, which is the dual concept of cartesian product, as follows:

$$\coprod_{\lambda \in \Lambda} X_\lambda = \bigcup_{\lambda \in \Lambda} X_\lambda \times \{\lambda\} .$$

Given $\mu \in \Lambda$, we define the *inclusion* $i_\mu : X_\mu \longrightarrow \coprod_{\lambda \longrightarrow \Lambda} X_\lambda$ by $i_\mu(x) = (x, \mu)$. For simplicity, it is usual to omit the second component (one can replace from the start each set X_λ with $X_\lambda \times \{\lambda\}$ so that they are disjoint from the start).

The disjoint union has the following universal property, which is dual to 4.3.1.

4.4.1 **Theorem**. *Given a family* $\{X_\lambda \mid \lambda \in \Lambda\}$ *of sets, the disjoint union* $\coprod_{\lambda\in\Lambda} X_\lambda$ *together with the inclusions* $i_\mu : X_\mu \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda$, $\mu \in \Lambda$, *is characterized by the following property:*

(DU) *Given any functions* $f_\lambda : X_\lambda \longrightarrow Y$, $\lambda \in \Lambda$, *there is a unique function* $f : \coprod_{\lambda\in\Lambda} X_\lambda \longrightarrow Y$ *such that* $f \circ i_\lambda \circ f = f_\lambda$. *In a diagram*

$$\begin{array}{ccc} X_\lambda & \xrightarrow{\ f_\lambda\ } & Y \\ {\scriptstyle i_\lambda}\big\downarrow & \nearrow_{f} & \\ \coprod_{\lambda\in\Lambda} X_\lambda\,. & & \end{array}$$

Proof: Given $(x,\lambda) \in \coprod_{\lambda\in\Lambda} X_\lambda$, define $f(x,\lambda) = f_\lambda(x)$. This is well defined since $x \in X_\lambda$).

If there are a set X and functions $j_\lambda : X_\lambda \longrightarrow X$ which fulfill (DU), namely such that given any functions $f_\lambda : X_\lambda \longrightarrow Y$, there is a function $f : X \longrightarrow Y$ such that $f \circ j_\lambda = f_\lambda$, then there is a bijective function $\psi : \coprod_{\lambda\in\Lambda} X_\lambda \longrightarrow X$ such that $\psi \circ i_\lambda = j_\lambda$. This can be easily verified using (DU) for both families of functions i_λ and j_λ. We leave the details as an *exercise* to the reader. □

We now pass to the topological case, and assume that we have a family $\{X_\lambda \mid \lambda \in \Lambda\}$ of topological spaces. We wish to furnish $\coprod_{\lambda\in\Lambda} X_\lambda$ with a topology such that the following two conditions hold:

(i) The inclusion $i_\lambda : X_\lambda \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda$ is continuous for all $\lambda \in \Lambda$.

(ii) Given a continuous map $f_\lambda : X_\lambda \longrightarrow Y$ for each $\lambda \in \Lambda$, the function $f : \coprod_{\lambda\in\Lambda} X_\lambda \longrightarrow Y$ such that $f \circ i_\lambda = f_\lambda$ is also continuous.

First we deal with condition (i) considering the following question: Given a set X and for each $\lambda \in \Lambda$ a function $i_\lambda : X_\lambda \longrightarrow X$, which is the finest topology in X that makes every i_λ continuous? This topology is clearly the infimum of all topologies in X coinduced by each i_λ. We shall particularly consider the following case. Let $X = \coprod_{\lambda\in\Lambda} X_\lambda$ be the disjoint union of the spaces X_λ taken as a set, and let $i_\lambda : X_\lambda \longrightarrow X$ be the inclusion. In this case, the question stated above gives rise to the problem that we want to study, that is, what is the maximal topology in the disjoint union such that the inclusions become continuous? This will lead us to the definition of the topological sum.

4.4.2 Definition. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a family of topological spaces and consider their disjoint union $X = \coprod_{\lambda\in\Lambda} X_\lambda$. Then the *sum topology* on X is the infimum of all topologies on X coinduced by the inclusions i_λ. This topology has as basis the collection of sets of the form

$$Q_\lambda \times \{\lambda\} \subset \coprod_{\lambda\in\Lambda} X_\lambda\,, \tag{4.4.3}$$

where Q_λ is open in X_λ, $\lambda \in \Lambda$.

Since the open sets in the sum topology are coinduced by all $i_\mu : X_\mu \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda$ and they are clearly of the form $\bigcup A_\lambda$ with $A_\lambda = \emptyset$ or $A_\lambda = X_\lambda$, $\lambda \neq \mu$ and A_μ is open in X_μ, we have the next theorem, where we assume that each space X_λ is a subset of the topological sum $\coprod_{\lambda\in\Lambda} X_\lambda$.

4.4.4 **Proposition**. *Let $\{X_\lambda\}$ be a family of topological spaces. A set A is open in the sum topology of $\coprod_{\lambda\in\Lambda} X_\lambda$ if and only if $A \cap X_\lambda$ is open in X_λ for every $\lambda \in \Lambda$. Thus the topology of the topological sum is such that the relative topology on each X_λ is the given topology.* □

The previous proposition is equivalent to the following.

4.4.5 **Proposition**. *Let $\{X_\lambda\}$ be a family of topological spaces. A set A is closed in the sum topology of $\coprod_{\lambda\in\Lambda} X_\lambda$ if and only if $A \cap X_\lambda$ is closed in X_λ for each $\lambda \in \Lambda$.* □

As an immediate consequence of 4.4.4 and 4.4.5 we obtain the following.

4.4.6 **Corollary**. *Every X_λ is open and closed in $\coprod_{\lambda\in\Lambda} X_\lambda$.* □

NOTATION. In case that Λ is finite, say $\Lambda = \{1,\ldots,k\}$, one usually denotes $\coprod_{\lambda\in\Lambda} X_\lambda$ simply by $X_1 \sqcup \cdots \sqcup X_k$.

4.4.7 REMARK. For the sake of simplicity, given a topological sum $\coprod_{\lambda\in\Lambda} X_\lambda$ we shall consider the *summands* X_λ strictly as subspaces of the topological sum.

The previous results make condition (i) clear, namely, that each $i_\lambda : X_\lambda \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda$ is continuous. We now prove that the sum topology also satisfies condition (ii).

4.4.8 **Proposition**. *Let X_λ, $\lambda \in \Lambda$, be a family of topological spaces and let $f_\lambda : X_\lambda \longrightarrow Y$ be a family of continuous maps. Then the unique map $f : \coprod_{\lambda\in\Lambda} X_\lambda \longrightarrow Y$ such that $f \circ i_\lambda = f_\lambda$, is continuous.*

Proof: Take an open set $B \subseteq Y$ and consider $f^{-1}(B) \subset \coprod_{\lambda\in\Lambda} X_\lambda$. In order to see if it is open we must take the inverse images under each i_λ. But $i_\lambda^{-1} f^{-1}(B) = (f \circ i_\lambda)^{-1}(B) = f_\lambda^{-1}(B)$. Since each f_λ is continuous, this inverse image is an open set, and thus $f^{-1}(B)$ is open. So f is continuous. □

4.4.9 Example. Any discrete space is the topological sum of all its points.

4.4.10 **Theorem**. *Let X be a topological space such that $X = \bigcup X_\lambda$ with $X_\lambda \cap X_\mu = \emptyset$ if $\lambda \neq \mu$. Then X is the topological sum of the spaces X_λ if and only if each X_λ is open in X.* □

The version for closed sets is valid only if Λ is finite.

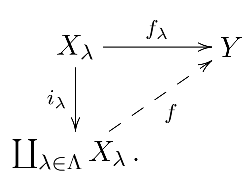
4.4.11 **Theorem**. *Let Λ be finite and let X be a topological space such that $X = \bigcup X_\lambda$ with $X_\lambda \cap X_\mu = \emptyset$ if $\lambda \neq \mu$. Then X is the topological sum of the spaces X_λ if and only if each X_λ is closed in X.* □

4.4.12 Note. If Λ is infinite, then the theorem is false. Namely, take $\Lambda = \mathbb{R}$. Then $\mathbb{R}$ is union of all its points, that is, $\mathbb{R} = \bigcup_{x\in\mathbb{R}}\{x\}$. The singular sets $\{x\}$ are closed sets, however $\mathbb{R}$ is not the topological sum of them, since since otherwise it would be discrete.

Dually to 4.3.16, we have the following universal property that characterizes the topological sum.

4.4.13 **Theorem**. *The topological sum $\coprod_{\lambda\in\Lambda} X_\lambda$ together with its inclusions $i_\lambda : X_\lambda \longrightarrow \coprod_{\lambda\in\Lambda} X_\lambda$ is characterized by the fact that the inclusions are continuous for all λ and the following property.*

(TS) *For every family $\{f_\lambda : X_\lambda \longrightarrow Y \mid \lambda \in \Lambda\}$ of continuous maps, there exists a unique continuous map $f : \coprod_{\lambda\in\Lambda} X_\lambda \longrightarrow Y$ such that $f|_{X_\lambda} = f \circ i_\lambda = f_\lambda$. In a diagram,*



$$f \text{ is continuous} \Leftrightarrow f_\lambda \text{ is continuous } \forall\lambda$$

The *proof* is a straightforward dualization of the proof of Theorem 4.3.16 and is left to the reader. □

4.4.14 EXAMPLES.

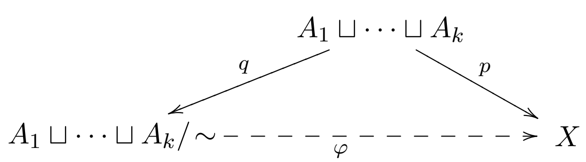
(a) A union of disjoint open intervals or a finite union of disjoint closed intervals in $\mathbb{R}$ is a topological sum.

(b) $\mathbb{Q} = \coprod_{k\in\mathbb{Z}}(k+t, k+1+t)_{\mathbb{Q}}$, where t is irrational and $(a,b)_{\mathbb{Q}} = (a,b)\cap\mathbb{Q}$, $a, b \in \mathbb{R}$. In other words $\mathbb{Q}$ can be expressed in very many ways as a topological sum of (open) intervals. However, $\mathbb{Q}$ is not the topological sum of all its points, otherwise it would be discrete, and it is not.

Take a topological space X and a finite family of closed subspaces $A_1, ..., A_k$ that cover X. Call $\iota_i : A_i \hookrightarrow X$ the inclusion map. Notice that ι_i is a closed map. One can recover X from the subspaces as follows.

4.4.15 **Proposition**. *Consider the topological sum $A_1 \sqcup \cdots \sqcup A_k$ and define an equivalence relation such that $A_i \ni x \sim y \in A_j$ if and only if $\iota_i(x) = \iota_j(y) \in X$. Then the quotient space $X' = A_1 \sqcup \cdots \sqcup A_k/\sim$ is homeomorphic to X.*

Proof: Consider the map $p : A_1 \sqcup \cdots \sqcup A_k \longrightarrow X$ such that $p|_{A_i} = \iota_i$. The map is clearly surjective, since the sets A_i form a cover. Furthermore, p is a closed map. Namely, if $C \subset A_1 \sqcup \cdots \sqcup A_k$ is closed, then $C_i = C \cap A_i$ is closed in A_i. One has that $p(C) = \iota_1(C_1) \cup \cdots \cup \iota_k(C_k)$, which is a closed set, since ι_i is a closed map and thus $\iota_i(C_i) \subset X$ is closed. Hence p is an identification. One clearly has the following commutative diagram:



The map φ is clearly bijective and since both p and q are identifications, φ is a homeomorphism. □

For open subspaces, one has a similar result. The finiteness assumption is not necessary in this case. If $\{A_\lambda\}$ is an arbitrary family of open subspaces that cover X, then one may recover X. Call $\iota_\lambda : A_\lambda \hookrightarrow X$ the inclusion map. Notice that ι_λ is an open map. The following can be proved similarly to the previous one.

4.4.16 **Proposition**. *Consider the topological sum $\coprod_\lambda A_\lambda$ and define an equivalence relation such that $A_\lambda \ni x \sim y \in A_\mu$ if and only if $\iota_\lambda(x) = \iota_\mu(y) \in X$. Then the quotient space $X' = \coprod_\lambda A_\lambda/\sim$ is homeomorphic to X.* □

Consider the *direct system* of maps of topological spaces

$$X^1 \xrightarrow{j_1^2} X^2 \xrightarrow{j_2^3} X^3 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{j_n^{n+1}} X^{n+1} \longrightarrow \cdots \tag{4.4.17}$$

and take the following quotient of the topological sum

$$X = \coprod_n X^n/\sim ,$$

where $X^n \ni x \sim j_n^{n+1}(x) \in X^{n+1}$. Let $q : \coprod_n X^n \longrightarrow X$ be the quotient map.

4.4.18 **Proposition**. *A subset $A \subset X$ is closed if and only if $q^{-1}(A) \cap X^n$ is closed for every n.*

Proof: The subset A is closed in X if and only if $q^{-1}(A)$ is closed in $\coprod_n X^n$, and this in turn is true if and only if $q^{-1}(A) \cap X^n$ is closed en X^n for every n. □

4.4.19 Definition. The space X defined above is called the *colimit* of the direct system (4.4.17) and is denoted by $\operatorname{colim}_n X^n$.† This space is furnished with maps $j_n \colon X^n \longrightarrow \operatorname{colim}_n X^n$ defined as the composites of the canonical inclusions into the disjoint union and the quotient map, namely

$$j_n \colon X^n \hookrightarrow \coprod_n X^n \overset{q}{\twoheadrightarrow} \operatorname{colim}_n X^n .$$

These maps satisfy $j_m \circ j_n^m = j_n \colon X^n \longrightarrow \operatorname{colim}_n X^n$, where $j_n^m = j_{m-1}^m \circ \cdots \circ j_n^{n+1} \colon X^n \longrightarrow X^m$, $m > n$.

The colimit has the following universal property which, as usual, characterizes it.

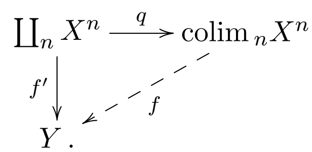
4.4.20 **Theorem**. *Let $\{f_n \colon X^n \longrightarrow Y \mid n \geq 1\}$ be a family of maps such that $f_{n+1} \circ j_n^{n+1} = f_n \colon X^n \longrightarrow Y$ for every $n \geq 1$ or, equivalently, $f_m \circ j_n^m = f_n \colon X^n \longrightarrow Y$ for every $m > n \geq 1$, then there exists a unique map $f : \operatorname{colim}_n X^n \longrightarrow Y$ such that $f \circ j_n = f_n$. In a diagram*

$$\begin{array}{ccc} X^n & \xrightarrow{j_n} & \operatorname{colim}_n X^n \\ {\scriptstyle f_n}\big\downarrow & \swarrow {\scriptstyle f} & \\ Y . & & \end{array}$$

f is continuous $\Leftrightarrow$ f_n is continuous $\forall n$.

†Some authors call this the *direct limit* and denote it by $\varinjlim X^n$ or dir lim X^n.

Proof: By the universal property of the topological sum 4.4.13, there exists a unique map $f' : \coprod_n X^n \longrightarrow Y$ such that $f'|_{X^n} = f_n$. Since $f_{n+1} \circ j_n^{n+1} = f_n$, $f'(j_n^{n+1}(x^n)) = f'(x^n)$. Thus, by the universal property of the quotient 4.2.22, there exists a unique $f : \coprod_n X^n/\sim = \operatorname{colim}_n X^n \longrightarrow Y$ such that the following diagram commutes:



This is the desired map f. □

4.4.21 Exercise. Prove that the property of the space $\operatorname{colim}_n X^n$ expressed in 4.4.20 characterizes the colimit.

4.4.22 Exercise. Prove that if one has a direct system

$$X^1 \xrightarrow{j_1^2} X^2 \xrightarrow{j_2^3} X^3 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{j_n^{n+1}} X^{n+1} \longrightarrow \cdots,$$

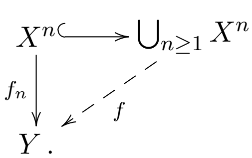
then the topology of $\operatorname{colim}_n X^n$ is the finest that makes the maps $j_n\colon X^n \longrightarrow \operatorname{colim}_n X^n$ continuous. (*Hint:* If $X = \operatorname{colim}_n X^n$ has a finer topology that makes the maps j_n continuous, then by 4.4.20, $\mathrm{id} : \operatorname{colim}_n X^n \longrightarrow X$ is continuous, and therefore the topology of X is also coarser than the colimit topology of $\operatorname{colim}_n X^n$.)

If we have in particular that

$$X^1 \subset X^2 \subset X^3 \subset \cdots \subset X^n \subset X^{n+1} \subset \cdots$$

is a chain of closed inclusions of topological spaces, we define their *union*, $\bigcup_{n\geq 1} X^n$, as the union of the sets X^n and its topology by declaring a subset $C \subset \bigcup_{n\geq 1} X^n$ as closed if and only if its intersection $C \cap X^n$ is closed in X^n for every $n \geq 1$. This topology will be called the *topology of the union*; it is also frequently named the *weak topology* with respect to the subspaces. This is obviously the colimit topology with respect to the inclusion maps.

4.4.23 Exercise. Prove that the union has the following universal property. Given a family of continuous maps $\{f_n\colon X^n \longrightarrow Y \mid n \geq 0\}$ such that $f_{n+1}|_{X^n} = f_n\colon X^n \longrightarrow Y$, then there exists a unique map $f\colon \bigcup X^n \longrightarrow Y$ such that $f|_{X^n} = f_n\colon X^n \longrightarrow Y$. In a commutative diagram we write this as



Conclude that, in this case, $\bigcup_n X^n \approx \operatorname{colim}_n X^n$ and show an explicit homeomorphism.

4.4.24 Examples.

(a) If one takes the chain of canonical inclusions of the Euclidean spaces

$$\mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \cdots \hookrightarrow \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \hookrightarrow \cdots ,$$

then one defines the *infinite dimensional Euclidean space* $\mathbb{R}^\infty$, as the union $\bigcup_n \mathbb{R}^n$ with the topology of the union. This space consists of *almost null* sequences $(x_n)_{n\in\mathbb{N}}$; that is, sequences such that for some n_0, $x_n = 0$ if $n \geq n_0$.

(b) The inclusions of (a) induce a chain of inclusions of spheres

$$\mathbb{S}^0 \hookrightarrow \mathbb{S}^1 \hookrightarrow \mathbb{S}^2 \hookrightarrow \cdots \hookrightarrow \mathbb{S}^n \hookrightarrow \mathbb{S}^{n+1} \hookrightarrow \cdots .$$

One defines the *infinite dimensional sphere* $\mathbb{S}^\infty$, as the union $\bigcup_n \mathbb{S}^n$ with the topology of the union. $\mathbb{S}^\infty$ is a subspace of $\mathbb{R}^\infty$ (*exercise*).

(c) If we take the projective spaces defined in 4.2.17(d), the inclusions of (b) induce a chain of inclusions

$$\{*\} = \mathbb{RP}^0 \hookrightarrow \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2 \hookrightarrow \cdots \hookrightarrow \mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1} \hookrightarrow \cdots .$$

We define the *infinite dimensional real projective space* $\mathbb{RP}^\infty$ as the union $\bigcup_n \mathbb{RP}^n$ with the topology of the union.

(d) Similarly to (a), we may take the chain of canonical inclusions of the complex spaces

$$\mathbb{C}^1 \hookrightarrow \mathbb{C}^2 \hookrightarrow \mathbb{C}^3 \hookrightarrow \cdots \hookrightarrow \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1} \hookrightarrow \cdots ,$$

then one defines the *infinite dimensional complex space* $\mathbb{C}^\infty$, as the union $\bigcup_n \mathbb{C}^n$ with the topology of the union. Again, this space consists of *almost null* sequences $(z_n)_{n\in\mathbb{N}}$. Since topologically $\mathbb{C}^n = \mathbb{R}^{2n}$, it is an easy *exercise* to prove that $\mathbb{C}^\infty = \mathbb{R}^\infty$.

(e) Similarly to (b), the inclusions of (d) induce a chain of inclusions of spheres

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \hookrightarrow \mathbb{S}^5 \hookrightarrow \cdots \hookrightarrow \mathbb{S}^{2n-1} \hookrightarrow \mathbb{S}^{2n+1} \hookrightarrow \cdots .$$

The union space of this chain is again the infinite dimensional sphere $\mathbb{S}^\infty$ (*exercise*).

(f) If we take the projective spaces defined in 4.2.17(f), the inclusions of (e) induce a chain of inclusions

$$\{*\} = \mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \hookrightarrow \cdots \hookrightarrow \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n \hookrightarrow \cdots .$$

We define the *infinite dimensional complex projective space* $\mathbb{CP}^\infty$ as the union $\bigcup_n \mathbb{CP}^n$ with the topology of the union.

4.4.25 EXERCISE. Since the inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ are inclusions of vector spaces, the space $\mathbb{R}^\infty$ clearly has a canonical vector space structure. Moreover, the infinite dimensional sphere is a subspace of it; that is, $\mathbb{S}^\infty \subset \mathbb{R}^\infty$, and thus, it makes sense to consider the element $-x$ for each $x \in \mathbb{S}^\infty$. Besides, if $x \in \mathbb{S}^\infty$, then $-x \in \mathbb{S}^\infty$. Declaring an equivalence relation in $\mathbb{S}^\infty$ by $x \sim -x$, prove that

$$\mathbb{RP}^\infty = \mathbb{S}^\infty/\!\sim \ .$$

4.4.26 EXERCISE. Since the inclusions $\mathbb{C}^n \subset \mathbb{C}^{n+1}$ are inclusions of complex vector spaces, the space $\mathbb{C}^\infty$ clearly has a canonical complex vector space structure. Moreover, as in the previous exercise, the infinite dimensional sphere is a subspace of it; that is, $\mathbb{S}^\infty \subset \mathbb{C}^\infty$, and thus, it makes sense to consider the element λx for each $x \in \mathbb{S}^\infty$ and $\lambda \in \mathbb{S}^1$. Besides, if $x \in \mathbb{S}^\infty$, then $\lambda x \in \mathbb{S}^\infty$. Declaring an equivalence relation in $\mathbb{S}^\infty$ by $x \sim \lambda x$, $\lambda \in \mathbb{S}^1$, prove that

$$\mathbb{CP}^\infty = \mathbb{S}^\infty/\!\sim \ .$$

Conclude that there is a canonical map

$$\mathbb{RP}^\infty \longrightarrow \mathbb{CP}^\infty \,.$$

Below, in Chapter 5, we shall analyze the more general concept of a colimit of a diagram of topological spaces.

4.4.27 EXERCISE. Consider the topological sum of two copies of the halfspace $\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and identify the subspaces $\mathbb{R}^{n-1}$ of each copy via the identity. Show that the quotient space obtained is homeomorphic to $\mathbb{R}^n$.

Chapter 5 Limits and colimits

The concepts of limit and colimit play a very important role in topology, since many of the relevant spaces are constructed as limits or colimits of given diagrams of spaces. We have already analyzed some special cases in 4.3.27 and 4.4.19 and many others that have been studied throughout this chapter are, indeed, limits or colimits.

In this chapter we shall study the most general cases of limits and colimits of diagrams of topological spaces and we shall obtain as special cases the already analyzed cases.

5.1 Diagrams

5.1.1 Definition. A *digraph*, or *directed graph*, consists of a set of *vertices* $V = \{a, b, c, \dots\}$ and a set of *oriented edges* $A = \{\alpha, \beta, \gamma, \dots\}$. In other words, there are two functions dom, cod $: A \longrightarrow V$. We call $\mathrm{dom}(\alpha)$ the *domain* of the edge α and $\mathrm{cod}(\alpha)$ the *codomain* of α, and if $\mathrm{dom}(\alpha) = a$ and $\mathrm{cod}(\alpha) = b$, we write $\alpha : a \longrightarrow b$, so that the digraph D can be visualized as shown in Figure 5.1.

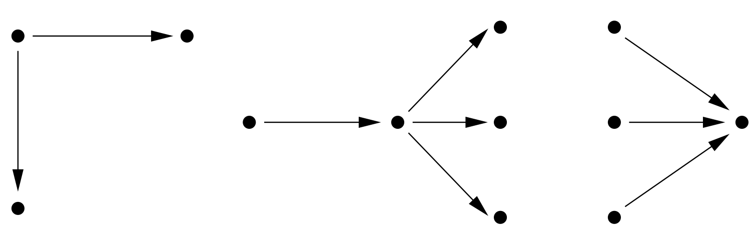


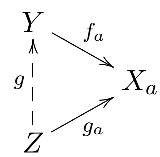
Figure 5.1 Digraphs

Digraphs will be useful to model diagrams of topological spaces.

5.1.2 Definition. Let D be a digraph with vertices V and oriented edges A. A *D-diagram*, or a *diagram modelled by D*, is a collection of topological spaces $\{X_a \mid a \in V\}$ and for each oriented edge $\alpha : a \longrightarrow b$ of A, a continuous map $h_\alpha : X_a \longrightarrow X_b$.

Hence, a D-diagram is a diagram of topological spaces arranged according to the geometry of D.

5.1.3 Definition. Let D be a digraph with vertices V and edges A. A D-*source* consists of a topological space Y, a D-diagram of spaces $\mathcal{X} = \{X_a, h_\alpha \mid a \in V, \alpha \in A\}$, and for each $a \in V$, maps $f_a : Y \longrightarrow X_a$ such that for each edge $\alpha \in A$, $\alpha : a \longrightarrow b$, $h_\alpha \circ f_a = f_b$. We denote the D-source by $\mathbf{f} : Y \longrightarrow \mathcal{X}$. We say that the D-source is *final* if for any other D-source $\mathbf{g} : Z \longrightarrow \mathcal{X}$, there exists a unique map $g : Z \longrightarrow Y$ such that for each vertex $a \in V$, $f_a \circ g = g_a$, namely, such that the diagram



is commutative for every $a \in V$.

Let $\mathcal{X}$ be a D-diagram and take $X' = \prod_{a\in V} X_a$ together with the projections $p_a : X' \longrightarrow X_a$. In general, it is not true that if $\alpha : a \longrightarrow b$ is an edge in D, then $h_\alpha \circ p_a = p_b$; hence, we have to consider the subspace X de X' such that

$$X = \{(x_a) \in \prod_{a\in V} X_a \mid h_\alpha(a) = b \,\forall\, \alpha : a \longrightarrow b,\, \alpha \in A\}\,.$$

Then the maps $h_a : X \longrightarrow X_a$ given by $h_a = p_a|_X$ determine a D-source $\mathbf{h} : X \longrightarrow \mathcal{X}$.

5.1.4 **Proposition**. *The D-source $\mathbf{h} : X \longrightarrow \mathcal{X}$ just constructed is final.*

Proof: Let $\mathbf{f} : Y \longrightarrow \mathcal{X}$ be any D-source; that is, one has maps $f_a : Y \longrightarrow X_a$. So we may define a unique map $f' : Y \longrightarrow X' = \prod_{a\in V} X_a$ such that $p_a \circ f' = f_a$, where p_a is the projection onto the a-factor; however, since $h_\alpha \circ f_a = f_b$, one has for every $y \in Y$ that $h_\alpha(f_a(y)) = f_b(y)$; that is, the point $(f_a(y))_{a\in V}$ of the product lies in fact on X; in other words, $y \mapsto (f_a(y))_{a\in V}$ determines a unique map

$$f : Y \longrightarrow X$$

such that $h_a \circ f = p_a \circ f' = f_a : Y \longrightarrow X_a$. This shows that the D-source $\mathbf{h} : X \longrightarrow \mathcal{X}$ is final. □

5.2 Limits

5.2.1 Definition. Let $\mathcal{X}$ be a D-diagram of topological spaces; a final D-source $\mathbf{h} : X \longrightarrow \mathcal{X}$ is called *limit** of the D-diagram and is denoted by $\mathbf{h} : \lim \mathcal{X} \longrightarrow \mathcal{X}$

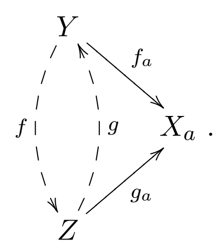
*some authors call this the *inverse limit* or $\mathbf{h} : \lim_D X_a \longrightarrow \mathcal{X}$. For simplicity, one says also limit to the space $\lim \mathcal{X}$.

In 5.1.4 we proved the there always exists a limit. In fact, the limit is unique and is characterized by the *universal property* of being final.

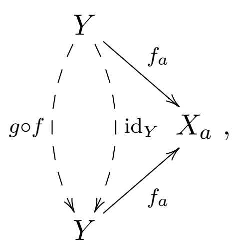
5.2.2 **Theorem**. *Let $\mathcal{X}$ be a D-diagram. Then its limit $\mathbf{h} : \lim \mathcal{X} \longrightarrow \mathcal{X}$ exists and is characterized uniquely by the fact of being a final D-source.*

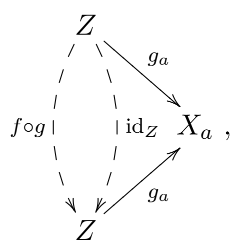
Proof: The existence was already established in 5.1.4. We see now that any two final D-sources $\mathbf{f} : Y \longrightarrow \mathcal{X}$ and $\mathbf{g} : Z \longrightarrow \mathcal{X}$ are *isomorphic*.

Since $\mathbf{f} : Y \longrightarrow \mathcal{X}$ is final, there exists a unique map $g : Z \longrightarrow Y$ such that $f_a \circ g = g_a$ for every $a \in V$. Similarly, since $\mathbf{g} : Z \longrightarrow \mathcal{X}$ is final, there exists a unique map $f : Y \longrightarrow Z$ such that $g_a \circ f = f_a$ for every $a \in V$; in a diagram



Since the following diagrams commute with the composite and the identity, respectively,





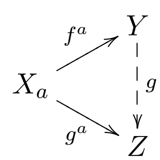
by the uniqueness of the the maps in the final sources, one has that $g \circ f = \mathrm{id}_Y$ and $f \circ g = \mathrm{id}_Z$. □

5.2.3 Exercise. Prove that the topology of $\lim \mathcal{X}$ is the coarsest that makes the maps $h_a : \lim \mathcal{X} \longrightarrow X_a$, $a \in V$, continuous.

5.3 Colimits

Throughout Chapter 4, we have been tacitly considering a kind of *duality* between different concepts: subspaces vs. quotients, topological products vs. topological sums, limits of inverse systems vs colimits of direct systems, etcetera. In this section we just analyzed the concept of limit of a D-diagram, which by definition was a certain final D-source. Its dual concept will be that of a colimit of a D-diagram, for which we require, in the first place, to define the dual concept of that of a final D-source.

5.3.1 DEFINITION. Let D be a digraph with vertices V and edges A. A D-*sink* consists of a D-diagram of spaces $\mathcal{X} = \{X_a, h_\alpha \mid a \in V, \alpha \in A\}$, a topological space Y, and for each $a \in V$, maps $f^a : X_a \longrightarrow Y$ such that for each edge $\alpha \in A$, $\alpha : a \longrightarrow b$, $f^b \circ h_\alpha = f^a$. We denote the D-sink by $\mathbf{f} : \mathcal{X} \longrightarrow Y$. We say that the D-sink is *initial* if for any other D-sink $\mathbf{g} : \mathcal{X} \longrightarrow Z$, there exists a unique map $g : Y \longrightarrow Z$ such that for each vertex $a \in V$, $g \circ f^a = g^a$, namely, such that the diagram



is commutative for every $a \in V$.

Let $\mathcal{X}$ be a D-diagram and take $X' = \coprod_{a \in V} X_a$ together with the inclusions $j^a : X_a \longrightarrow X'$. In general, it is not true that if $\alpha : a \longrightarrow b$ is an edge in D, then $j^b \circ h_\alpha = j^a$. Hence we have to consider the quotient space X of X' such that

$$X = \coprod_{a \in V} X_a / \sim ,$$

where $j^a(x_a) \sim j^b(h_\alpha(x_a))$, for every edge $\alpha : a \longrightarrow b$, $\alpha \in A$. Let $q : X' \longrightarrow X$ be the quotient map. Then the maps $h^a : X_a \longrightarrow X$ given by $h^a = q \circ j^a$, determine a D-sink $\mathbf{h} : \mathcal{X} \longrightarrow X$.

5.3.2 **Proposition**. *The D-sink $\mathbf{h} : \mathcal{X} \longrightarrow X$ just constructed is initial.*

Proof: Let $\mathbf{f} : \mathcal{X} \longrightarrow Y$ be any D-sink, that is one has maps $f^a : X_a \longrightarrow Y$. So we may define a unique map $f' : X' = \coprod_{a \in V} X_a \longrightarrow Y$ such that $f' \circ i^a = f^a$, where i^a is the inclusion into the a-summand. However, since $f^b \circ h_\alpha = f^a$, one has for every $x_a \in X_a$ and every a that $f^b(h_\alpha(x_a)) = f^a(x_a)$, namely, f' identifies x_a with $h_\alpha(x_a)$, so that it factors through q, and hence, it determines a unique map

$$f : X \longrightarrow Y$$

such that $f \circ h^a = f' \circ i^a = f^a : X_a \longrightarrow Y$. This shows that the D-sink $\mathbf{h} : \mathcal{X} \longrightarrow X$ is initial. $\square$

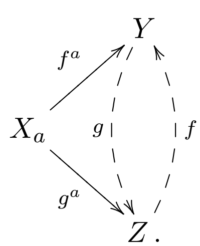
5.3.3 Definition. Let $\mathcal{X}$ be a D-diagram of topological spaces. An initial D-sink $\mathbf{h} : \mathcal{X} \longrightarrow X$ is called *colimit*[†] of the D-diagram and is denoted by $\mathbf{h} : \mathcal{X} \longrightarrow \operatorname{colim} \mathcal{X}$ or $\mathbf{h} : \mathcal{X} \longrightarrow \operatorname{colim}_D X_a$. For simplicity, one says also colimit to the space $\operatorname{colim} \mathcal{X}$.

In 5.3.2 we proved that there always exists a colimit. In fact, the colimit is unique and is characterized by the *universal property* of being initial.

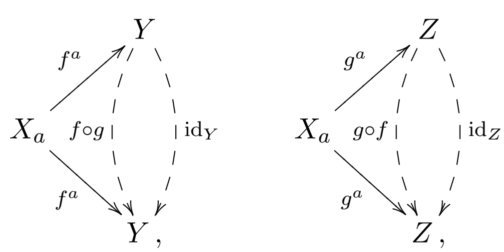
5.3.4 **Theorem**. *Let $\mathcal{X}$ be a D-diagram. Then its colimit $\mathbf{h} : \mathcal{X} \longrightarrow \operatorname{colim} \mathcal{X}$ exists and characterized uniquely by the fact of being an initial D-sink.*

Proof: The existence was already established in 5.1.4. We see now that any two D-initial sinks $\mathbf{f} : \mathcal{X} \longrightarrow Y$ and $\mathbf{g} : \mathcal{X} \longrightarrow Z$ are *isomorphic*.

Since $\mathbf{f} : \mathcal{X} \longrightarrow Y$ is initial, there exists a unique map $g : Y \longrightarrow Z$ such that $g \circ f^a = g^a$ for every $a \in V$. Similarly, since $\mathbf{g} : \mathcal{X} \longrightarrow Z$ is initial, there exists a unique map $f : Z \longrightarrow Y$ such that $f \circ g^a = f^a$ for every $a \in V$, in a diagram



Since the following diagrams commute with the composite and the identity, respectively,



by the uniqueness of the maps in the initial sinks, one has that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_Y$. □

5.3.5 Exercise. Prove that the topology of $\operatorname{colim} \mathcal{X}$ is the finest that makes the maps $h^a : X_a \longrightarrow \operatorname{colim} \mathcal{X}$, $a \in V$, continuous.

[†]some authors call this the *direct limit*

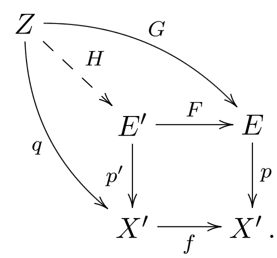
5.3.6 DEFINITION. Let $f : X' \longrightarrow X$ and $p : E \longrightarrow X$ be continuous maps. We define the space $E' = \{(x', e) \mid f(x') = p(e)\} \subset X' \times E$, with the topology as a subspace of the topological product, and the maps $F : E' \longrightarrow E$ and $p' : E' \longrightarrow X'$ given by $F = \mathrm{proj}_{X'}|_{E'}$ and $p' = \mathrm{proj}_E|_{E'}$. The commutative diagram

$$\begin{array}{ccc} E' & \xrightarrow{F} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow[f]{} & X' \end{array}$$

is called *cartesian square* or *pullback diagram* (Cf. 8.4.19.)

5.3.7 EXERCISE. Prove that the triple $(E'; p', F)$ is characterized uniquely by the following universal property.

(PB) If $q : Z \longrightarrow X'$ and $G : Z \longrightarrow E$ are maps such that $p \circ G = f \circ q$, then there exists a unique map $H : Z \longrightarrow E'$ such that $p' \circ H = q$ and $F \circ H = G$. In a diagram



In other words, the triple $(E'; p', F)$ is universal among the triples $(Z; q, G)$. Conclude that $(E'; p', F)$ is the limit of the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow[f]{} & X'\,. \end{array}$$

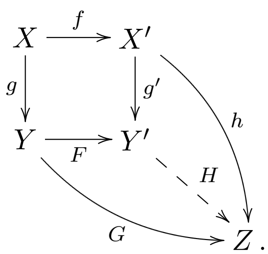
5.3.8 DEFINITION. Let $f : X \longrightarrow X'$ and $g : X \longrightarrow Y$ be continuous maps. We define the space $Y' = Y \sqcup X'/{\sim}$, where $g(x) \sim f(x)$, with the topology as a quotient space of the topological sum, and the maps $F : Y \longrightarrow Y'$ and $g' : X' \longrightarrow Y'$ given by $F = q \circ i_Y$ and $g = q \circ i_{X'}$, where $q : Y \sqcup X' \longrightarrow Y'$ is the quotient map and i_Y, $i_{X'}$ are the corresponding inclusions. The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow[F]{} & Y' \end{array}$$

is called *cocartesian square* or *pushout diagram* (Cf. with the definition of a double attaching space 5.4.1 at the start of the following section.)

5.3.9 Exercise. Prove that the triple $(F, g'; Y')$ is characterized uniquely by the following universal property.

(PO) If $G : Y \longrightarrow Z$ and $h : X' \longrightarrow Z$ are maps such that $h \circ f = G \circ g$, then there exists a unique map $H : Y' \longrightarrow Z$ such that $H \circ F = G$ and $H \circ g' = h$. In a diagram



In other words, the triple $(F, g'; Y')$ is universal among the triples $(G, h; Z)$. Conclude that $(F, g'; Y')$ is the colimit of the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle g}\downarrow & & \\ Y\,. & & \end{array}$$

5.4 Special constructions

Take the digraph

$$D: \qquad b \xleftarrow{\alpha} a \xrightarrow{\beta} c$$

and let $\mathcal{X}$ be any D-diagram, say

$$X_b \xleftarrow{f_\alpha} X_a \xrightarrow{f_\beta} X_c\,.$$

For simplicity, we call $A = X_a$, $X = X_b$ and $Y = X_c$. Besides, $f = f_\alpha$ and $g = f_\beta$, namely, we have the diagram

$$\mathcal{X}: \qquad X \xleftarrow{f} A \xrightarrow{g} Y\,.$$

5.4.1 Definition. The *double attaching space* is defined as the space $\operatorname{colim} \mathcal{X}$, that we denote by $Y_g \cup_f X$, and schematically looks as shown in Figure 5.2. That is,

$$Y_g \cup_f X = X \sqcup Y / f(a) \sim g(a)\,, \quad a \in A\,.$$

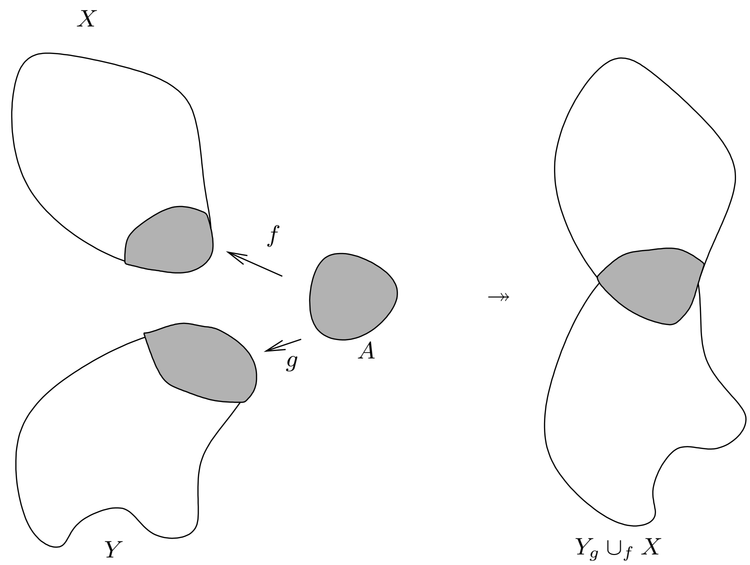


Figure 5.2 The double attaching space

5.4.2 Remark. The double attaching space is precisely the construction that we made in 5.3.8, at the end of the previous section to define the concept of a cocartesian square. For that reason, $Y_g \cup_f X$ can be called the *pushout* of (f, g).

5.4.3 Examples.

(a) If $A \subset Y$ and $g : A \hookrightarrow Y$ is the inclusion, then $Y_g \cup_f X$ is called the *attaching space* of $f : A \longrightarrow X$ and is denoted simply by $Y \cup_f X$.

(b) If $A = \emptyset$, then $Y_g \cup_f X = X \sqcup Y$.

(c) If $A \subset X$, $Y = \{*\}$, and $f : A \hookrightarrow X$ is the inclusion map, then $Y_g \cup_f X = X/A$.

5.4.4 Exercise. Consider the maps $X \xleftarrow{f} A \xrightarrow{g} Y$ and their double attaching space $Y_g \cup_f X$. Show that $Y_g \cup_f X$ is indeed a pushout, namely show that it is characterized by the following property: Given maps $\varphi : X \longrightarrow Z$ and $\psi : Y \longrightarrow Z$ such that $\varphi \circ f = \psi \circ g$, there exists a unique map $\xi : Y_g \cup_f X \longrightarrow Z$ such that $\xi \circ i = \varphi$ and $\xi \circ j = \psi$, where $i : X \longrightarrow Y_g \cup_f X$ and $j : Y \longrightarrow Y_g \cup_f X$ are the inclusions. (Cf. Exercise 5.3.9.)

The construction of the attaching space is a very important construction, since with it many associated constructions can be obtained, that play a relevant role in several branches of topology. Many of them are based in a particular space associated to a given topological space that we define in what follows.

5.4.5 DEFINITION. Let $f : X \longrightarrow Y$ be continuous and consider the diagram

$$X \times I \xleftarrow{i_0} X \xrightarrow{f} Y\,,$$

where $i_0(x) = (x, 0)$. The attaching space $Y \cup_f (X \times I)$ is called the *mapping cylinder* of f and is denoted by M_f.

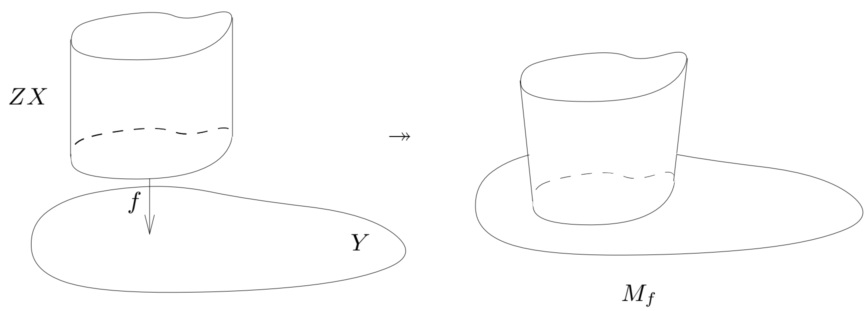


Figure 5.3 The mapping cylinder

If $Y = X$ and $f = \mathrm{id}_X$, then $M_f = X \times I$, and this space is called simply the *cylinder* over X and is denoted by ZX. The quotient space of the cylinder over X, in which the top is collapsed to a point, $ZX/X \times \{1\} = X \times I/X \times \{1\}$ is called the *cone* over X and is denoted by CX. There is a natural inclusion $X \hookrightarrow CX$ given by $x \mapsto q(x, 0)$, where $q : ZX \longrightarrow CX$ is the quotient map.

5.4.6 DEFINITION. Let $f : X \longrightarrow Y$ be continuous and consider the diagram

$$CX \hookleftarrow X \xrightarrow{f} Y\,.$$

The attaching space $Y \cup_f CX$ is called the *mapping cone* of f and is denoted by C_f.

5.4.7 EXERCISE. Prove that $C\mathbb{S}^{n-1} \approx \mathbb{B}^n$.

In what follows, we define a new space obtained from a given space, that plays an important role in algebraic topology, mainly in homotopy theory.

5.4.8 DEFINITION. Take $Y = *$. The mapping cone of $f : X \longrightarrow *$, namely $* \cup_f CX$, is called the *suspension* of X and is denoted by ΣX.

5.4.9 EXERCISE. Prove that the suspension of a space X coincides with the quotient of the cylinder $ZX = X \times I$ obtained by collapsing the *top* of the cylinder $X \times \{1\}$ into a point, and the *bottom* of the cylinder $X \times \{0\}$ into another point.

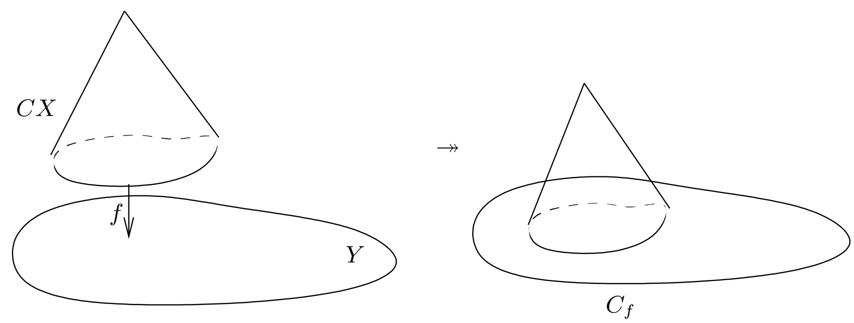


Figure 5.4 The mapping cone

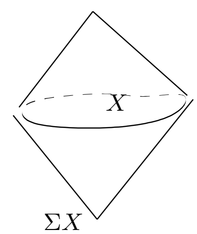


Figure 5.5 The suspension

5.4.10 EXERCISE. Prove that the suspension of a space X is homeomorphic to the double attaching space corresponding to the diagram

$$CX \hookleftarrow X \hookrightarrow CX\ ,$$

or, equivalently, to the mapping cone of the inclusion $X \hookrightarrow CX$.

Another interesting construction is the following.

5.4.11 DEFINITION. Let $f : X \longrightarrow X$ be continuous. The quotient space of the cylinder over X, ZX, obtained after identifying the bottom of the cylinder with the top through the identification $(x, 0) \sim (f(x), 1)$, is called the *mapping torus* of f and is denoted by T_f.

5.4.12 EXAMPLES.

(a) If $f = \mathrm{id}_X : X \longrightarrow X$, then the mapping torus T_f is called *torus* of X and is denoted by TX.

(b) If $X = I$, then TI is the *trivial band* or *standard cylinder*.

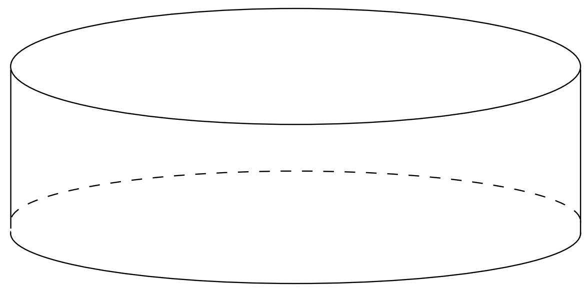


Figure 5.6 The trivial band

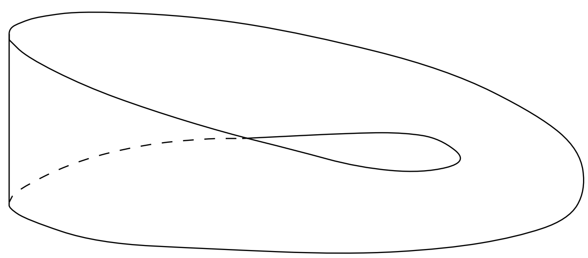


Figure 5.7 The Moebius band

(c) If $X = I$, and $f : I \longrightarrow I$ is such that $f(t) = 1 - t$, then T_f is the *Moebius band*.

(d) If $X = \mathbb{S}^1$, then $T\mathbb{S}^1$ is the *standard torus* and is denoted by $\mathbb{T}^2$.

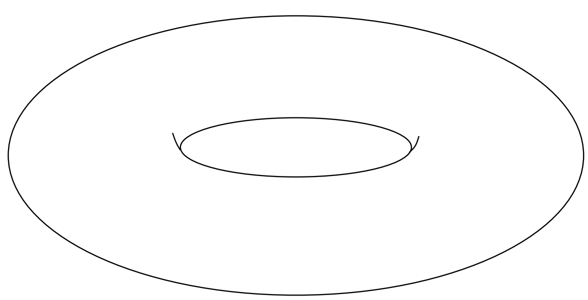


Figure 5.8 The standard torus

(e) If $X = \mathbb{S}^1 = \{\zeta = \mathrm{e}^{2\pi \mathrm{i}t} \in \mathbb{C} \mid t \in I\}$ and $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is such that $f(\mathrm{e}^{2\pi \mathrm{i}t}) = \mathrm{e}^{-2\pi \mathrm{i}t}$, then T_f is the *Klein bottle*.

5.4.13 Exercise. Prove that the trivial band is homeomorphic to the space obtained from the *square* $I \times I$ by identifying each point $(s, 0)$ with $(s, 1)$, while

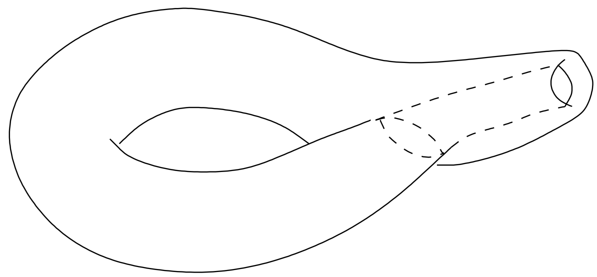


Figure 5.9 The Klein bottle

the Moebius band is homeomorphic to the space obtained from the square by identifying $(s, 0)$ with $(1 - s, 1)$.

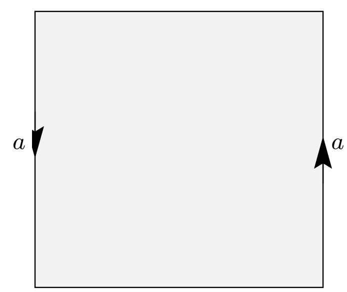


Figure 5.10 The construction of the Moebius band

5.4.14 Exercise. Prove that the standard torus is homeomorphic to the space obtained from the square $I \times I$ by identifying the points $(s, 0)$ with $(s, 1)$, and $(0, t)$ with $(1, t)$, while the Klein bottle is obtained from the square by identifying $(s, 0)$ with $(1-s, 1)$, and $(0, t)$ with $(1, t)$. This is the classical definition of the bottle. We shall see below in 5.4.20 a different way of defining it.

5.4.15 Exercise. Prove that the Klein bottle is a quotient space of the Moebius band. Describe the identification.

5.4.16 Exercise. Let $q : \mathbb{S}^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ be the quotient map such that $q(x) = q(-x)$, and let

$$\mathbb{B}^n \xleftarrow{\ i\ } \mathbb{S}^{n-1} \xrightarrow{\ q\ } \mathbb{RP}^{n-1} .$$

Prove that $\mathbb{RP}^{n-1} \cup_q \mathbb{B}^n \approx \mathbb{RP}^n$.

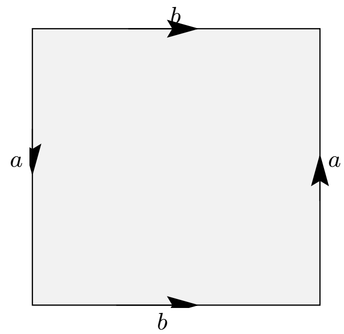


Figure 5.11 The construction of the Klein bottle

5.4.17 Definition. Consider the diagram

$$\mathbb{B}^n \xleftarrow{\ i\ } \mathbb{S}^{n-1} \xrightarrow{\ \varphi\ } X \ .$$

The resulting attaching space $X \cup_\varphi \mathbb{B}^n$ is said to be obtained by *attaching a cell* of dimension n to the space X. Frequently this space is denoted by $X \cup_\varphi e^n$. The image of $\mathbb{B}^n$, resp. $\overset{\circ}{\mathbb{B}}{}^n$, in $Y = X \cup_\varphi \mathbb{B}^n$ is called *closed cell*, resp. *open cell* de Y, and φ is called *characteristic map* of the cell of Y.

5.4.18 Exercise. Prove that $\mathbb{RP}^n$ is obtained from $\mathbb{RP}^{n-1}$ by attaching a cell of dimension n with the canonical map $q : \mathbb{S}^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ as characteristic map. Conclude that the projective space $\mathbb{RP}^n$ is obtained by successively attaching cells of dimensions $1, 2, \dots, n$ to the singular space $\{*\}$; that is,

$$\mathbb{RP}^n = \{*\} \cup e^1 \cup e^2 \cup \cdots \cup e^n \,.$$

(*Hint:* $\mathbb{RP}^n$ is obtained from $\mathbb{B}^n$ by identifying in its boundary $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ each pair of antipodal points in one point.)

5.4.19 Exercise. Take $\mathbb{B}^n \hookleftarrow \mathbb{S}^{n-1} \hookrightarrow \mathbb{B}^n$. Prove that the corresponding attaching space is homeomorphic to $\mathbb{S}^n$, namely, $\mathbb{S}^n$ is obtained from $\mathbb{B}^n$ by attaching a cell of dimension n with the inclusion as characteristic map. Equivalently, $\mathbb{S}^n$ is obtained from $\mathbb{S}^{n-1}$ by attaching two cells of dimension n, each with $\mathrm{id}_{\mathbb{S}^{n-1}}$ as characteristic map. Conclude that the n-sphere can be decomposed as

$$\mathbb{S}^n = \mathbb{S}^0 \cup (e_1^1 \cup e_2^1) \cup (e_1^2 \cup e_2^2) \cup \cdots \cup (e_1^n \cup e_2^n)\,.$$

The procedure of decomposing a space and then putting it together again shown in Propositions 4.4.15 and 4.4.16 is called *cutting and pasting*.

5.4.20 Examples.

1. Consider the torus $\mathbb{T}^2$ embedded in 3-space in such a way that it is symmetric with respect to the origin (namely, so that $x \in \mathbb{T}^2$ if and only if $-x \in \mathbb{T}^2$, see Figure 5.12 (a)). We shall show using the method of cutting and pasting, that if we identify each point $x \in \mathbb{T}^2$ with its opposite $-x$, then we obtain the Klein bottle. The result of this identification is the same as if we slice the torus along the inner and outer equators and keep the upper (closed) part, so that we obtain a ring (see Figure 5.12 (b)), and then identify with each other antipodal points of the outer circle and those of the inner circle. This is marked with double and single arrows. Now cut the ring into two halves along the dotted lines. Up to a homeomorphism we obtain two rectangles, where we use different types of arrows to codify what has to be identified (see Figure 5.12 (c)). Flip the top rectangle (Figure 5.12 (d)) and identify the edges marked with the single solid arrow (Figure 5.12 (e)). We obtain a square such that after realizing the identifications marked therein, we obtain the Klein Bottle (cf. Exercise 5.4.14).

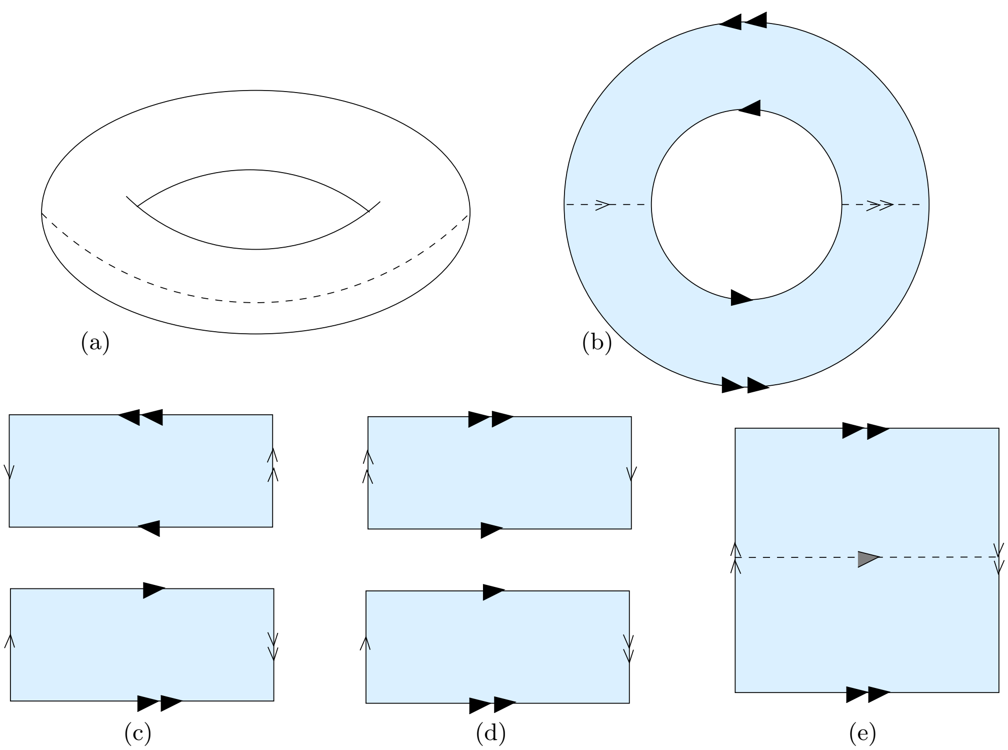


Figure 5.12 Cutting and pasting the torus

2. Consider the square $I \times I$ and identify on the boundary the points of the form $(1, t)$ with $(1 - t, 1)$, and those of the form $(0, 1 - t)$ with $(t, 0)$. This is illustrated in Figure 5.13 (a), where edge a is identified with edge a' and edge b with edge b', preserving the counterclockwise orientation (cf. 4.2.25

(c)). Now cut the square along the diagonal form $(0,0)$ to $(1,1)$. Calling the new edges c and c', one obtains two triangles with sides a, b, c and a', b', c' oriented as shown in Figure 5.13 (b). We may now glue both triangles along the edges a and a' preserving the orientations (turning and flipping the first triangle), and so we obtain a quadrilateral with edges b, c, b', c' shown in Figure 5.13 (c). Clearly this quadrilateral is homeomorphic (via an affine map) to a square with edges equally denoted shown in Figure 5.13 (d). Now observe that in the resulting square one identifies the vertical sides with the same orientation, namely $(0,t)$ with $(1,t)$, and the horizontal edges with the opposite direction, namely $(t,0)$ with $(1-t,1)$. The result of this last identification is the Klein bottle defined in 5.4.14.

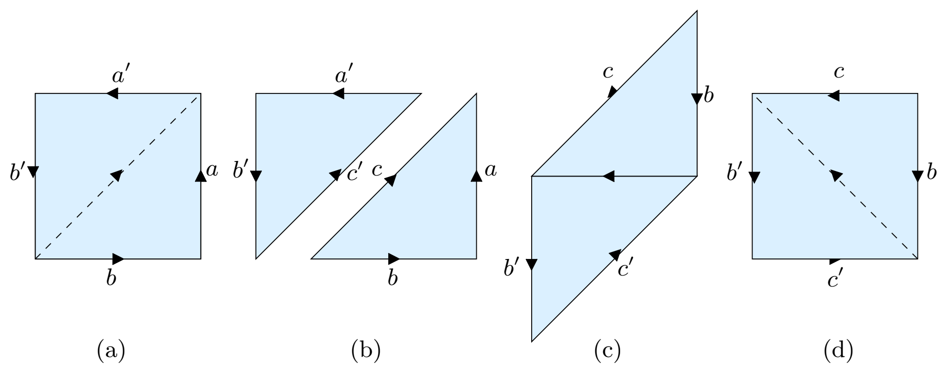


Figure 5.13 Cutting and pasting the Klein bottle

5.4.21 Exercise. Using the method of cutting and pasting, show that the result of cutting the Moebius band along the equator (cf. Example 6.1.39) yields the trivial band, namely a space homeomorphic to $\mathbb{S}^1 \times I$. Notice that if you realize that in the 3-space, one obtains a band with a double twisting, as shown on Figure 5.14.

5.5 Group actions

An interesting aspect of topology is the one that links it with the algebra. This relationship appears in different ways. An important way refers to what we may call the symmetry of a topological space, that brings us directly to the origins of group theory itself, because initially the groups were precisely conceived as symmetry groups.

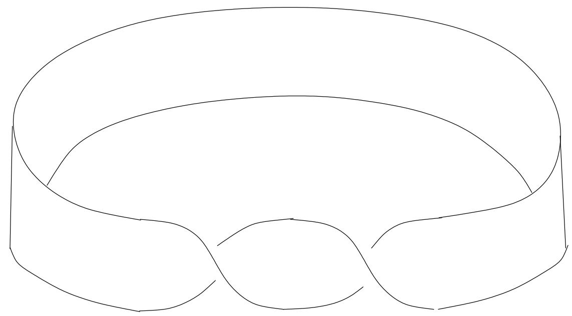


Figure 5.14 Cutting the Moebius band yields the trivial band

5.5.1 Definition. Let G be a finite group. We say that G *acts* on a topological space X if there is a continuous map

$$\alpha : G \times X \longrightarrow X\,,$$

called the *action*, with G considered as a discrete space such that, if we denote $\alpha(g, x)$ simply by gx, the following identities hold:

$$x = x$$
$$g_1(g_2x) = (g_1g_2)x\,,$$

where in the last identity, the action of the group is applied twice on the left hand side, while on the right hand side it is applied only once after multiplying inside the group.

5.5.2 Exercise. Prove that given $g \in G$, the map $t_g : X \longrightarrow X$ such that $t_g(x) = gx$ is a homeomorphism. This map is called *translation* in X by the element g.

As the previous exercise shows, the action of the group corresponds to a set of homeomorphisms of the space onto itself, with the group structure given by composition. This suggests the symmetry of the space.

5.5.3 Example. Let $G = \mathbb{Z}_2$ be the group with two elements $\{1, -1\}$ and let $X = \mathbb{S}^n$. Then one has a group action

$$\mathbb{Z}_2 \times \mathbb{S}^n \longrightarrow \mathbb{S}^n$$

such that $(-1)x = -x$. This action is called *antipodal action* on the n-sphere.

A finite group determines a digraph with only one vertex and an edge for each element of the group, as shown in Figure 5.15.

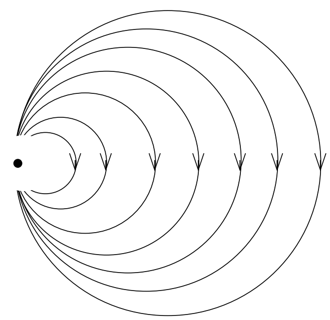


Figure 5.15 A group seen as a digraph

Therefore, an action of the group on a space X determines a diagram, as shown in Figure 5.16, that we call *G-space*. Hence Example 5.5.3 is a $\mathbb{Z}_2$-space.

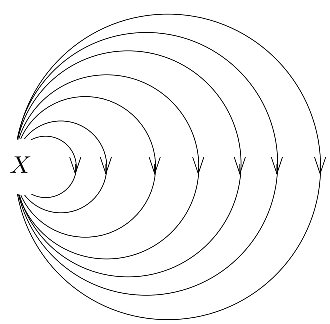


Figure 5.16 The action of a group on a space X

5.5.4 Definition. The colimit of a G-space X is called the *quotient of X under the action of G*, or more simply, the *orbit space* of the G-space X. This colimit is usually denoted by X/G.

5.5.5 Exercise. Prove that X/G is the quotient space de X obtained by identifying in X the point x with the point gx for every element $g \in G$. The set $Gx = \{gx \in X \mid g \in G\}$ is called the *orbit* of the point x in the G-space X. Hence, X/G is obtained by identifying every orbit of X in one point.

5.5.6 Examples.

(a) Under the antipodal action on $\mathbb{S}^n$ (5.5.3), the orbits in $\mathbb{S}^n$ are the pairs of points $\{x, -x\}$ and the quotient of $\mathbb{S}^n$ under the antipodal action $\mathbb{S}^n/\mathbb{Z}_2$ is $\mathbb{RP}^n$.

(b) The group $\mathbb{Z}$ (with $+$ as group multiplication) acts on $\mathbb{R}$ so that if $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, $\alpha(n, x) = x + n$. Similarly, $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ by $\alpha((n_1, n_2), (x_1, x_2)) = (x_1 + n_1, x_2 + n_2)$

5.5.7 EXERCISE. Prove that $\mathbb{R}/\mathbb{Z} \approx \mathbb{S}^1$ (*Hint:* The map $\mathbb{R} \longrightarrow \mathbb{S}^1$ such that $t \mapsto \mathrm{e}^{2\pi \mathrm{i} t}$ determines the homeomorphism.)

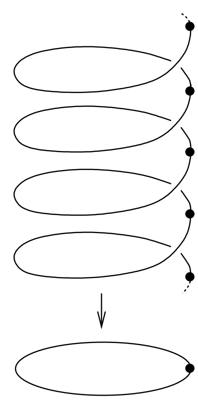


Figure 5.17 The identification $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$

5.5.8 EXERCISE. Prove that $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z}) \approx \mathbb{T}^2$.

5.5.9 EXERCISE. Prove that there is an action of $\mathbb{Z}$ on $\mathbb{R}^2$ such that

$$(n, (x_1, x_2)) \mapsto (n + x_1, (-1)^n x_2)\,.$$

Prove, moreover, that the orbit space of this action, $\mathbb{R}^2/\mathbb{Z}$, is homeomorphic to the Moebius band.

Chapter 6 Connectedness

Connectedness is a fundamental concept in topology. Notwithstanding its simplicity, it has deep consequences. Related to (topological) connectedness, there are the concepts of path-connectedness and their local versions. In this chapter we shall study all these properties of a topological space. We shall consider many examples. The real line is an example of a space having all kinds of connectedness. We shall show with examples that path-connectedness is a strictly stronger property than connectedness, as well as local path-connectedness is strictly stronger than local connectedness. Using the connectedness property, we shall prove the intermediate value theorem, which is a valuable tool in calculus.

6.1 Connected spaces

We infer from Examples 4.4.14 that $\mathbb{R}$ cannot be expressed as a disjoint union of open sets, but we did show that $\mathbb{Q}$ could be so expressed. This phenomenon points out an essential difference between the topology of $\mathbb{Q}$ and the topology of $\mathbb{R}$. This difference is more evident if we compare, say, $\mathbb{R}$ with a discrete space with more than one element. It would be possible to say that, in a way, $\mathbb{R}$ is "continuous," while the other space is "discrete." The difference lies in the concept of connectedness.

6.1.1 Definition. Let X be a topological space. We say that X is *disconnected* if there exist disjoint nonempty open sets A and B in X such that $X = A \cup B$. The pair (A, B) is called a *disconnection** of X. The space X is *connected* if it is not disconnected.

The following is a nice characterization of disconnectedness.

6.1.2 **Proposition**. *A topological space X is disconnected if and only if there exists a continuous surjective map $g : X \longrightarrow \{-1, 1\}$, where $\{-1, 1\}$ has the discrete topology.*

*Some authors call this a *separation* of X.

Proof: Assume first that X is disconnected and let (A, B) be a disconnection of X. Just define g by $g|_A = 1$ and $g|_B = -1$. Since $\{-1, 1\}$ is discrete, it is enough to notice that the inverse images $g^{-1}(1) = A$ and $g^{-1}(-1) = B$ are open. □

6.1.3 **Proposition**. *A topological space X is connected if and only if $\emptyset$ and X are the only subsets of X which are simultaneously open and closed.*

Proof: Let X be connected. If $A \subset X$ is open and closed, then $B = X - A$ is also open and closed. Since X is connected, if $A \neq \emptyset$, then $B = \emptyset$ and so $A = X$. Conversely, if $\emptyset$ and X are the only subsets of X which are simultaneously open and closed, and A and B are open and disjoint and their union is X, then one is the complement of the other. Hence A and B are also closed. Thus one is $\emptyset$ and the other is X. Therefore X is connected. □

Connectedness is obviously a topological invariant, namely, if X is connected, then any other space homeomorphic to X is connected.

A subset C of X is disconnected if and only if there exist open subsets A and B of X such that $(A \cap C, B \cap C)$ is a disconnection of the subspace C. The pair (A, B) is called a *disconnection* of C in X (see Figure 6.1).

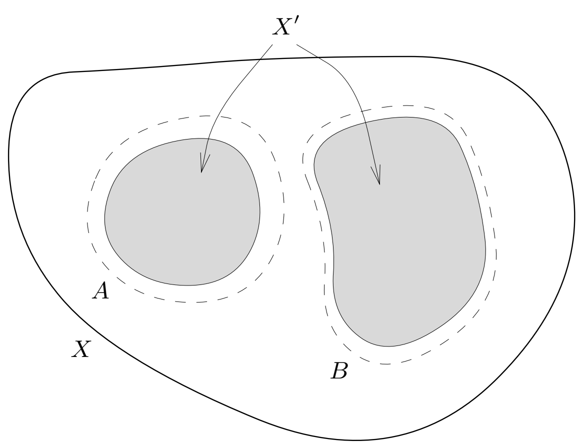


Figure 6.1 A disconnection of a subspace

6.1.4 REMARK. In other words, a disconnection (A, B) of a subset $C \subseteq X$ is a pair of subsets A, B of X such that

(a) $A, B \subset X$ are open.

(b) $A \cap C \neq \emptyset \neq B \cap C$.

(c) $A \cap B \cap C = \emptyset$.

(d) $C \subset A \cup B$.

We have the following result.

6.1.5 **Proposition**. *Let (A, B) be a disconnection of a space X and take $C \subset X$. Then (A, B) is a disconnection of C in X if $A \cap C \neq \emptyset \neq B \cap C$.* □

The next result will be useful below.

6.1.6 **Proposition**. *Let (A, B) be a disconnection of a topological space X, and let $C \subset X$ be a connected subset. Then either $C \subset A$ or $C \subset B$.*

Proof: Otherwise, (A, B) would be a disconnection of C in X. □

6.1.7 **Proposition**. *The following statements are equivalent:*

(a) *X is disconnected.*

(b) *There exist disjoint nonempty closed sets C and D in X such that $X = C \cup D$.*

(c) *There exists a proper nonempty subset A of X such that it is open and closed.*

□

In some sense, one might say that the paradigm of the connected spaces is $\mathbb{R}$. We have the following.

6.1.8 **Theorem**. *The space of the real numbers $\mathbb{R}$ is connected.*

We shall obtain this result from the more general one below 6.1.10. We start with the following.

6.1.9 Definition. Consider a totally ordered set R (see 3.4.2 below) with more than one element. We say that R is a *linear continuum* if the following hold:

(a) R has the *supremum property*, namely the property that every set, which is bounded from above, has a least upper bound, i.e. a *supremum*.

(b) R has the *density property*, namely, for each two elements $a < b$ in R, there is c such that $a < c < b$.

6.1.10 **Theorem**. *If R is a linear continuum endowed with the order topology, then R is connected.*

Proof: Assume, on the contrary, that R is disconnected and let (A, B) be a disconnection. Choose $a \in A$ and $b \in B$ and for convenience assume that $a < b$. Consider the closed interval $[a, b]$, which is the union of the (relatively) open sets

$$A' = A \cap [a, b] \quad \text{and} \quad B' = B \cap [a, b]\,.$$

The sets A' and B' are nonempty since $a \in A'$ and $b \in B'$. Hence (A', B') is a disconnection of $[a, b]$.

Since A' is nonempty and bounded, by the supremum property, the point $c = \sup A'$ exists. We shall prove that c does not belong to either A' or B'. This will be a contradiction since $c \in [a, b]$ and $[a, b] = A' \cup B'$.

Case 1. Assume $c \in B'$. Then either $c = b$ or $a < c < b$. In either case, since B' is open in $[a, b]$, there is an interval of the form $(d, c]$ contained in B'. If $c = b$, then we arrive to a contradiction, since d would be a smaller upper bound of A' than c. If $c < b$, notice that $(c, b]$ does not meet A', since c is an upper bound of A'. Then we have that the interval $(d, b] = (d, c] \cup (c, b]$ does not meet A'. Then we have again that d would be a smaller upper bound of A' than c, contradicting the construction of c. See Figure 6.2 (a).

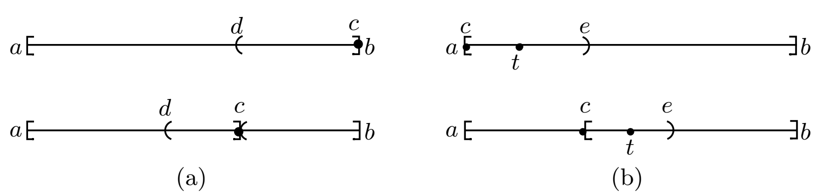


Figure 6.2 A linear continuum is connected

Case 2. Assume $c \in A'$. In this case $c \neq b$ and so either $c = a$ or $a < c < b$. Since A' is open in $[a, b]$, there is an interval of the form $[c, e)$ contained in A'. See Figure 6.2 (b). By the density property, there is a point $t \in R$ such that $c < t < e$. Hence $t \in A'$ and this contradicts the fact that c is an upper bound of A'. $\square$

Since clearly $\mathbb{R}$ is a linear continuum, it is a connected space. This is the *proof* of 6.1.8. $\square$

6.1.11 Exercise.

(a) Show that the interval $I = [0, 1]$ is connected.

(b) Show that the connected subspaces of $\mathbb{R}$ are the intervals, that is, the *bounded intervals* (a,b), $(a,b]$, $[a,b)$, $[a,b]$, for $a \leq b$ in $\mathbb{R}$, the *halflines* $(-\infty,b)$, $(-\infty,b]$, (a,∞), $[a,\infty)$, the line itself $\mathbb{R}$, and the empty interval $\emptyset$.

(c) Show that if $a < c < b$, then $[a,b]-\{c\}$, $(a,b]-\{c\}$, $[a,b)-\{c\}$, and $(a,b)-\{c\}$ are disconnected. Notice that each of a and b can be taken as $-\infty$ and ∞. In particular, $\mathbb{R}-\{0\}$ is disconnected.

(d) Show that no two of the intervals $(0,1)$, $(0,1]$, and $[0,1]$ are homeomorphic. (*Hint:* What happens if a point from each of these spaces is removed?) Conclude that no two of the intervals (a,b), $(c,d]$, and $[e,f]$, where $a<b$, $c<d$, and $e<f$, are homeomorphic.

Connectedness is preserved not only under homeomorphisms, but under any continuous maps. We have the following result.

6.1.12 **Theorem**. *Let X be a connected space. If $f: X \longrightarrow Y$ is a continuous map, then $f(X)$ is connected.*

Proof: Assume that $f(X)$ is disconnected and take a disconnection (A,B) of $f(X)$ in Y. Since f is continuous, the pair $(f^{-1}(A), f^{-1}(B))$ is a disconnection of X. □

6.1.13 Exercise. Do the converse statements of 6.1.12 hold? Namely:

(a) Let $f: X \longrightarrow Y$ be continuous and $f(X)$ connected. Is X connected?

(b) Let $f: X \longrightarrow Y$ be such that for every connected $C \subset X$, $f(C)$ is connected. Is f continuous?

6.1.14 Exercise. Show that $\mathbb{R}$ and $\mathbb{R}^n$ are homeomorphic if and only if $n=1$.

6.1.15 **Theorem**. *Let X be a topological space and for each $\lambda \in \Lambda$, take a connected subset $X_\lambda \subset X$ such that $X = \bigcup X_\lambda$. If there exists $\lambda_0 \in \Lambda$ such that $X_{\lambda_0} \cap X_\lambda \neq \emptyset$, then X is connected.*

Proof: Assume that X is disconnected. We shall see that if X_λ is connected for every $\lambda \neq \lambda_0$, then X_{λ_0} is disconnected. Let (A,B) be a disconnection of X. We prove that it is also a disconnection of X_{λ_0} in X. For this, it is enough to prove that $A \cap X_{\lambda_0} \neq \emptyset \neq B \cap X_{\lambda_0}$. Without losing generality, we may assume that $A \cap X_{\lambda_0} \neq \emptyset$. Since X_λ is connected for every $\lambda \neq \lambda_0$, by 6.1.6, either $X_\lambda \subset A$ or $X_\lambda \subset B$. Let $\Lambda_B = \{\lambda \in \Lambda \mid X_\lambda \subset B\}$. Then $\Lambda_B \neq \emptyset$, since if $\Lambda_B = \emptyset$, then $X_\lambda \subset A$ for every λ and $B = B \cap (\bigcup X_\lambda)$ would be empty. Take thus $\lambda \in \Lambda_B$. Consequently, $X_\lambda \subset B$ and $\emptyset \neq X_\lambda \cap X_{\lambda_0} \subset B \cap X_{\lambda_0}$. □

6.1.16 **Corollary**. *Let X be a topological space and let $X_\lambda \subset X$ be a family of connected sets, where $\lambda \in \Lambda$, such that $X = \bigcup X_\lambda$. If $\cap X_\lambda \neq \emptyset$, then X is connected.* □

6.1.17 EXERCISE. Let X_n be a sequence of connected subspaces of X such that for all n, $X_n \cap X_{n+1} \neq \emptyset$. Show that $\bigcup_n X_n$ is connected.

6.1.18 EXERCISE. Let X be a space such that each pair of points $x, y \in X$ lies inside some connected subset $K_{xy} \subseteq X$. Show that X is connected.

6.1.19 EXERCISE. Let X be an infinite set furnished with the cofinite topology. Show that it is connected.

6.1.20 EXERCISE. Recall the space $\mathbb{R}$ with the lower half-open interval topology, which we called $\mathbb{R}_l$ (3.4.13). Is it connected? Why?

6.1.21 EXERCISE. Let X be a connected space and let $Y \subset X$ be a connected subspace. Show that if (A, B) is a disconnection of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected sets.

6.1.22 **Theorem**. *Let X be a topological space and $C \subset X$. If C is connected, then its closure $\overline{C}$ is connected.*

Proof: If $\overline{C}$ is disconnected, take a disconnection (A, B) of $\overline{C}$ in X. Since A and B are open sets, one easily verifies that (A, B) satisfies conditions (a)–(d) for C in 6.1.4, specially (b) $A \cap C \neq \emptyset \neq B \cap C$. Hence (A, B) is a disconnection of C too. □

More generally, we have the following consequence.

6.1.23 **Corollary**. *Let X be a topological space and $C \subset X$. If C is connected and C' is such that $C \subset C' \subset \overline{C}$, then C' is connected.*

Proof: It is enough to observe that the closure of C in C' is C'. □

The connectedness property is inherited by finite products. We have the following.

6.1.24 **Proposition**. *If X_i, $i = 1, \ldots, n$, is a finite family of nonempty spaces, then their topological product $X_1 \times \cdots \times X_n$ is connected if and only if X_i is connected, $i = 1, \ldots, n$.*

Proof: We prove the case of two connected spaces X_1 and X_2 and we obtain the general case by induction. We fix a point $(x_1, x_2) \in X_1 \times X_2$. Since $X_1 \times \{x_2\}$ is homeomorphic to X_1, it is connected. Similarly, $\{x\} \times X_2$ is connected for all $x \in X_1$. The point (x, x_2) lies in the intersection $(X_1 \times \{x_2\}) \cap (\{x\} \times X_2)$. Therefore, by 6.1.16, the cross

$$T_x = (X_1 \times \{x_2\}) \cup (\{x\} \times X_2)$$

is connected. Notice that the chosen point (x_1, x_2) lies in T_x for all $x \in X_1$, hence $\bigcap_{x \in X_1} T_x \neq \emptyset$. Again by 6.1.16, $\bigcup_{x \in X_1} T_x$ is connected. This union clearly equals $X_1 \times X_2$, so that this product is connected.

Now we may inductively assume that $X_1 \times \cdots \times X_{n-1}$ is connected. Therefore, by the first case, $X_1 \times \cdots \times X_n = (X_1 \times \cdots \times X_{n-1}) \times X_n$ is connected.

Conversely, assume that the product $X = X_1 \times \cdots \times X_n$ is connected. Since X_i is the image of X under the ith projection, then, by Theorem 6.1.12, X_i is connected for all i.

The previous result is true for infinite products. Solve the next.

6.1.25 EXERCISE. Show that if X_λ, $\lambda \in \Lambda$, is a nonempty family of nonempty spaces, then their product $\prod_{\lambda \in \Lambda} X_\lambda$ is connected if and only if each of its factors X_λ is connected. (*Hint:* Fix a point $p = (p_\lambda) \in \prod_{\lambda \in \Lambda} X_\lambda$ and do the following:

(i) For a finite subset K of Λ, put $X_K = \{x \in \prod_{\lambda \in \Lambda} X_\lambda \mid x_\lambda = p_\lambda \forall \lambda \notin K\}$ and show that X_K is connected-

(ii) Show that $\bigcup_{K \subset \Lambda} X_K$ is connected.

(iii) Show that $\overline{\bigcup_{K \subset \Lambda} X_K} = \prod_{\lambda \in \Lambda} X_\lambda$ and conclude that $\prod_{\lambda \in \Lambda} X_\lambda$ is connected.)

6.1.26 EXERCISE. Put $\mathbb{R}_i = \mathbb{R}$ for $i = 1, 2, \ldots$, and take the cartesian product $\mathbb{R}^\omega = \prod_{i=1}^\infty \mathbb{R}_i$ endowed with the box topology (see Note 4.3.7). Show that $\mathbb{R}^\omega$ is disconnected. (*Hint:* Let A consist of all bounded sequences in $\mathbb{R}^\omega$ and let B consist of all unbounded sequences in $\mathbb{R}^\omega$. Show that A and B are open sets by proving that given a sequence $(x_i) \in \mathbb{R}^\omega$, the open set

$$A = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times \cdots$$

consists of bounded sequences if x is bounded and of unbounded sequences if x is unbounded. Conclude that A and B are open and therefore (A, B) is a disconnection of $\mathbb{R}^\omega$.)

There are two famous results in topology:

- **The Borsuk–Ulam theorem**: *Given any continuous map* $f : \mathbb{S}^n \longrightarrow \mathbb{R}^n$*, there is a point* $x \in \mathbb{S}^n$ *such that* $f(x) = f(-x)$*.*

- **The Brouwer fixed-point theorem**: *Given any continuous map* $f : \mathbb{D}^n \longrightarrow \mathbb{D}^n$*, there is a point* $x \in \mathbb{D}^n$ *such that* $f(x) = x$*.* This point x is called a *fixed point* of f.

Using connectedness, we may prove these two theorems in case $n = 1$, as well as the intermediate value theorem 6.1.27, using essentially the same proof. Indeed, we shall use the facts that the spaces $\mathbb{S}^1$ and $\mathbb{D}^1 = [-1, 1]$ are connected. Namely, $[-1, 1]$ is connected, since it is the closure of the open interval $(-1, 1)$, which is homeomorphic to $\mathbb{R}$, and $\mathbb{R}$ is connected. Since $\mathbb{S}^1$ is the continuous image of the interval $[0, 1]$ by the exponential map $t \mapsto \mathrm{e}^{2\pi\mathrm{i}t}$, then by Theorem 6.1.12, $\mathbb{S}^1$ is connected. We have the next.

6.1.27 **Theorem**. (Intermediate value) *Given a continuous map* $f : [a, b] \longrightarrow \mathbb{R}$ *such that* $f(a) \neq f(b)$*, and given a real number* r *between* $f(a)$ *and* $f(b)$*, there is a point* $x \in (a, b)$ *such that* $f(x) = r$*.*

6.1.28 **Theorem**. (Borsuk–Ulam) *Given any continuous map* $f : \mathbb{S}^1 \longrightarrow \mathbb{R}$*, there is a point* $x \in \mathbb{S}^1$ *such that* $f(x) = f(-x)$*.*

6.1.29 **Theorem**. (Brouwer) *Given any continuous map* $f : [-1, 1] \longrightarrow [-1, 1]$*, there is a point* $x \in [-1, 1]$ *such that* $f(x) = x$*.*

We give a common *proof* for all three results. We do this by contradiction. If either of the three results were false, one might define new maps

$$g_1 : [a, b] \longrightarrow \{-1, 1\}\,, \quad g_2 : \mathbb{S}^1 \longrightarrow \{-1, 1\}\,, \quad \text{or} \quad g_3 : [-1, 1] \longrightarrow \{-1, 1\}\,,$$

by

$$g_1(x) = \frac{f(x) - r}{|f(x) - r|}\,, \quad g_2(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}\,, \quad \text{or} \quad g_3(x) = \frac{f(x) - x}{|f(x) - x|}\,.$$

In either case, since the denominator is nonzero, the new map is continuous. In the first case, we may assume without loss of generality that $f(a) < r < f(b)$. Hence $g_1(a) = -1$ and $g_1(b) = 1$, so that g_1 is surjective. In the second case, we observe that $g_2(-x) = -g_2(x)$ (i.e., g_2 is a so-called *odd function*), so that it is also surjective. Finally, in the third case, $g_3(a) = -1$ and $g_3(b) = 1$, thus g_3 is surjective too. By 6.1.2, the existence of these maps contradicts the connectedness of $[a, b]$, $\mathbb{S}^1$, and $[a, b]$, respectively. □

6.1.30 EXERCISE. Show that the three previous theorems 6.1.27, 6.1.28, and 6.1.29 ($n = 1$) are equivalent, namely that each of them is a consequence of any of the other two.

6.1.31 EXERCISE. Prove the following "meteorological theorem": At any given moment in time, there are two antipodal points on the equator of the earth at which both the temperature and the barometric pressure are equal.

6.1.32 EXERCISE. Take a set X and consider two topologies $\mathcal{A} \subset \mathcal{A}'$ on X. What can be said about connectedness of X with respect to one topology and with the other?

6.1.33 NOTE. The one-point subset $\{x\} \subset X$ is obviously connected. The set $C_x = \bigcup\{C \subset X \mid x \in C, C \text{ connected}\}$ is connected by 6.1.16, and is the largest connected set that contains x. The subset C_x is called the *connected component* of X corresponding to x. By 6.1.22, C_x is necessarily closed. If $y \in C_x$, then $C_y = C_x$. Thus the sets C_x build a partition of X. However C_x is not necessarily open.

6.1.34 EXERCISE.

(a) Let $X = \mathbb{Q}$. Show that $C_x = \{x\}$ for every $x \in \mathbb{Q}$.

(b) Show that the converse of 6.1.22 does not hold by giving an example of a disconnected set in a topological space whose closure is connected.

(c) Give an example of a connected set whose interior is disconnected.

6.1.35 EXERCISE. Show that if a space X has only finitely many connected components, then each of these components is open in X. (We already know by 6.1.33 that they are closed.) Hence in this case, X is the topological sum of its components.

The number of connected components of a topological space is an *invariant* that allows to distinguish between spaces.

6.1.36 EXAMPLE. The finite sets seen as discrete topological spaces are classified by their cardinality, that is, $A \approx B$ if and only if $\#(A) = \#(B)$. In particular, $\mathbb{S}^0 \not\approx \mathbb{B}^0$.

The following is an example of how it is possible to obtain other invariants.

6.1.37 EXAMPLE. Let X be a topological space. If $x_0 \in X$, let $Z(x_0)$ be the number of connected components of $X - x_0$. If $\varphi : X \longrightarrow Y$ is a homeomorphism and $y_0 = \varphi(x_0)$, then φ determines a homeomorphism $X - x_0 \approx Y - y_0$; therefore, $Z(x_0) = Z(y_0)$. Hence, the invariant $Z(x_0)$ allows us to decide if two spaces are not homeomorphic. In particular, $I \not\approx \mathbb{S}^1$, since in I, $Z(\frac{1}{2}) = 2$, but for any point $x \in \mathbb{S}^1$, $Z(x) = 1$. Similarly, we may show that the cross X in Figure 6.3 is not homeomorphic to I, because for any point $x \in I$, $Z(x) \leq 2$; however, there exists a point O in X such that $Z(O) = 4$.

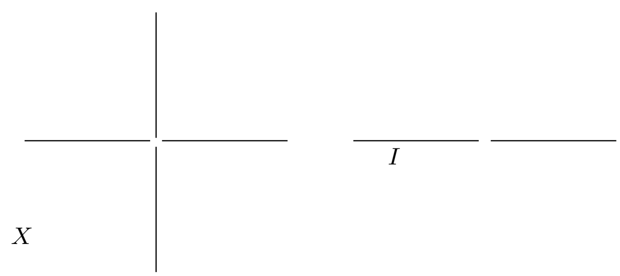


Figure 6.3 A cross is not homeomorphic to an interval

It is fun to solve the following exercise.

6.1.38 EXERCISE. Using the Z-invariant, classify up to homeomorphism, the letters of the alphabet seen as subspaces of $\mathbb{R}^2$.

A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

(In particular, $\mathsf{H} \not\approx \mathsf{T}$, because in H there are two points x such that $Z(x) = 3$, while in T there is only one.)

More interesting is the following example.

6.1.39 EXAMPLE. The (open) Moebius band and the (open) trivial band (see 5.4.12 below) are not homeomorphic, because if we remove the equator from the Moebius band, we obtain a connected space (namely a space with a single connected component), while removing any circle from the trivial band we always obtain a disconnected space (with two connected components).

6.1.40 EXERCISE. Let D and D' be two open disks in $\mathbb{R}^2$ whose closures $\overline{D}$ and $\overline{D}'$ intersect in exactly one point, so that the boundary circles of the two disks are tangent. Determine which of the following subspaces of $\mathbb{R}^2$ are connected:
(a) $D \cup D'$. (b) $\overline{D} \cup \overline{D}'$. (c) $\overline{D} \cup D'$.

6.1.41 Exercise. Show that the set $A \subset \mathbb{R}^2$ consisting of pairs (s, t) such that at least one of s and t is rational is connected.

6.1.42 Definition. A topological space X such that $C_x = \{x\}$ for every $x \in X$ is called *totally disconnected*.

6.1.43 Example. If X is discrete then it is totally disconnected. The converse is false, because by 6.1.34(a) $\mathbb{Q}$ is totally disconnected and is not discrete. (See also next exercise.)

6.1.44 Exercise. Let C be the Cantor set defined in 2.2.7.

(a) Show that C is totally disconnected.

(b) Show that C has no isolated points (see 2.3.6).

6.2 Locally connected spaces

6.2.1 Definition. We say that a space X is *locally connected* if the connected neighborhoods of each point in X build a local basis. Namely, X is locally connected if and only if for every $x \in X$ and for every neighborhood U of x, there exists a connected neighborhood V of x such that $V \subset U$.

6.2.2 Examples.

(a) $\mathbb{Q}$ is not locally connected.

(b) I and $\mathbb{R}$ are locally connected.

(c) Obviously, not every locally connected space is connected. Also, not every connected space is locally connected, namely the *infinite comb*, defined as the subspace of $\mathbb{R}^2$ given by

$$P = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, y > 0 \Leftrightarrow x = 0,\ \frac{1}{n},\ n \in \mathbb{N}\},$$

is connected (see Figure 6.4); however, it is not locally connected, since for any point $a = (0, y) \in P$, $y > 0$, any small enough neighborhood is disconnected. Namely, a neighborhood of each of these points in P looks as in Figure 6.5

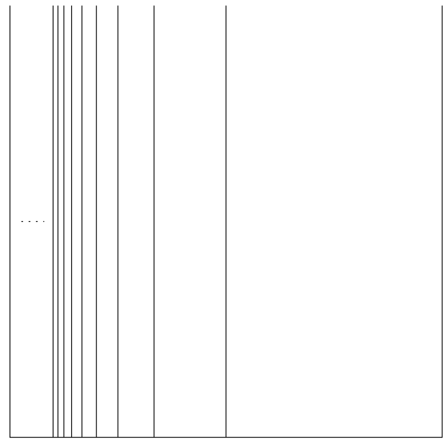


Figure 6.4 The infinite comb

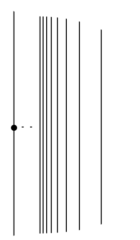


Figure 6.5 A neighborhood in the infinite comb

6.2.3 Exercise.

(a) Prove that if X is locally connected, then for every $x \in X$, its connected component C_x is open.

(b) Let X be the *topologist's sine curve* given by taking $A = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1, y = \sin(\frac{\pi}{x})\}$, $B = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$, and $X = A \cup B$. (See Figure 6.6.) Prove that X is connected. Is X locally connected?

6.2.4 Remark. By 6.1.33 and 6.2.3(a), we have that if X is locally connected, then C_x is open and closed. On the other hand, $C_x \cap C_y \neq \emptyset \Leftrightarrow C_x = C_y$, because otherwise $C_x \cup C_y$ would be a connected set that is strictly larger than C_x and C_y. Thus we can define an equivalence relation in X such that $x \sim y \Leftrightarrow C_x = C_y$ and call Λ the set of equivalence classes. For each $\lambda \in \Lambda$, take $C_\lambda = C_x$ for some

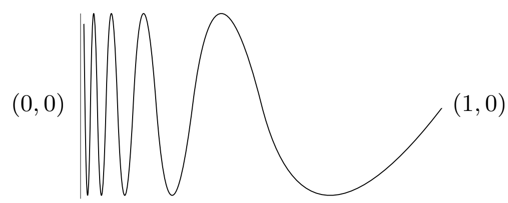


Figure 6.6 The topologist's sine curve

$x \in \lambda$. The subspaces $C_\lambda \subset X$ are called *connected components* of X. Hence, the connected components of a space X are the maximal connected sets in X.

By Remark 6.2.4 and by Theorem 4.4.10, we have the following.

6.2.5 **Theorem**. *If X is a locally connected space, then*

$$X = \coprod_{\lambda \in \Lambda} C_\lambda \,,$$

where $\{C_\lambda \mid \lambda \in \Lambda\}$ is the family of connected components de X. □

6.2.6 Exercise. If X is locally connected, then the equivalence relation defined in 6.2.4 determines a quotient map $X \longrightarrow \Lambda$. Prove that Λ with the quotient topology is discrete. Show with an example that if X is not locally connected, then the previous statement is false. (*Hint:* Take $X = \mathbb{Q}$.)

6.2.7 Exercise. Show that the Sorgenfrey line $\mathbb{E}$ (see 3.4.13) is not locally connected.

6.2.8 Exercise. Show that the continuous image of a locally connected space need not be not locally connected.

6.2.9 Exercise. Show that the nonempty product $\prod_{\lambda \in \Lambda} X_\lambda$ is a locally connected space if and only if the following hold:

(a) Each factor space X_λ is locally connected.

(b) All but finitely many factor spaces X_λ are connected.

6.3 PATH-CONNECTED SPACES

There is a stronger concept than that of connectedness, namely the so-called *path connectedness*†.

6.3.1 DEFINITION. A *path* in a topological space X is a map $\sigma : I \longrightarrow X$. The point $x_0 = \sigma(0)$ is called the *start* of the path and the point $x_1 = \sigma(1)$, is called the *end* of the path. We shall denote this fact by $\sigma : x_0 \simeq x_1$ and we shall also say that σ *joins* x_0 with x_1 or that x_0 and x_1 are connected by the path σ.

The following are paths which play an important role. The *constant path* $\kappa_x : x \sim x$ given by $\kappa_x(t) = x$ for all $t \in I$; given a path $\sigma : x_0 \simeq x_1$, the *inverse path* $\overline{\sigma} : x_1 \simeq x_0$ given by $\overline{\sigma}(t) = \sigma(1-t)$; and given two paths $\sigma : x_0 \simeq x_1$ and $\tau : x_1 \simeq x_2$, the *path product* $\sigma\tau : x_0 \simeq x_2$, defined by

$$\sigma\tau(t) = \begin{cases} \sigma(2t) & \text{if } t \leq \frac{1}{2}, \\ \tau(2t-1) & \text{if } \frac{1}{2} \leq t, \end{cases}$$

is a path from x to z.)

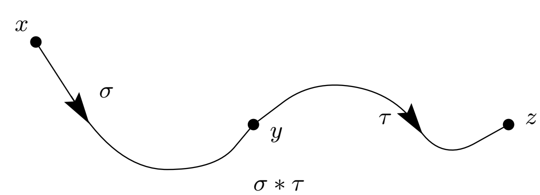


Figure 6.7 The path product

Using the constant, the inverse, and the product paths, one easily proves the next.

6.3.2 **Proposition**. *The relation* $x_0 \sim x_1 \Leftrightarrow \sigma : x_0 \simeq x_1$*, for some path* σ*, is an equivalence relation.* □

6.3.3 DEFINITION. The equivalence classes under the relation $\simeq$ are called the *path-components* of the space X and the set of equivalence classes, that is the set of path-components, is usually denoted by $\pi_0(X)$.

We say that a topological space $X \neq \emptyset$ is *path connected* if it has only one path-component, that is, if given any two points $x, y \in X$, then there exists a path that joins them.

†I like to call the "connectedness" concept, "*topological* connectedness", and the "path connectedness" concept, "*homotopical* connectedness".

6.3.4 Examples. The following are path-connected spaces:

(a) The interval I, and with it, any other interval in $\mathbb{R}$, including $\mathbb{R}$ itself.

(b) Any indiscrete space.

(c) The Sierpinski space (2.6.10).

6.3.5 Exercise. Check that in fact the previous are examples of path-connected spaces.

Clearly, the property of a space of being path connected is a topological invariant.

6.3.6 **Proposition**. *Every path-connected space X is connected.*

Proof: If X is not connected and x, $y \in X$ are in different connected components, say C_x and C_y, and if $\sigma : I \longrightarrow X$ is a path that joins x and y, then the inverse images $\sigma^{-1}C_x$ and $\sigma^{-1}C_y$ build a disconnection of I. This contradicts 6.1.11(a). $\square$

6.3.7 Example. The converse of 6.3.6 does not hold. For instance, the topologist's sine curve defined in 6.2.3(b), and given by

$$X = \overline{\left\{ \left(x, \sin\left(\frac{\pi}{x}\right)\right) \,\middle|\, x \in (0,1] \right\}}$$

is connected, because it is the closure of a connected set. But it is not path connected, since there is no path that joins $(0,0)$ and $(1,0)$ in X. See Figure 6.6.

Similarly to 6.1.12, we have the following result.

6.3.8 **Theorem**. *Let $f : X \longrightarrow Y$ be continuous and let X be path connected. Then $f(X)$ is path connected.*

Proof: Given $f(x)$, $f(y) \in f(X)$, take $\sigma : x \simeq y$; then $f \circ \sigma : f(x) \simeq f(y)$. $\square$

6.3.9 Remark. The space $\{(x, \sin(\frac{\pi}{x})) \mid x \in (0,1]\}$ is path connected, since it is the image of the interval $(0,1]$ under the continuous map given by $x \mapsto (x, \sin(\frac{\pi}{x}))$. Thus Example 6.3.7, in contrast with 6.1.22, also shows that the closure of a path-connected space need not be path connected.

Similarly to 6.1.15 and 6.1.16, we have the following two results.

6.3.10 **Theorem**. *Let X be a topological space and for each $\lambda \in \Lambda$, take a path connected subset $X_\lambda \subset X$ such that $X = \bigcup X_\lambda$. If there exists $\lambda_0 \in \Lambda$ such that $X_{\lambda_0} \cap X_\lambda \neq \emptyset$, then X is path connected.*

Proof: Take two points $x, y \in X$ and assume that $x \in X_\lambda$ and $y \in X_\mu$. Since $X_\lambda \cap X_{\lambda_0} \neq \emptyset$ and $X_\mu \cap X_{\lambda_0} \neq \emptyset$, we may take points $x' \in X_\lambda \cap X_{\lambda_0}$ and $y' \in X_\mu \cap X_{\lambda_0}$. Since X_λ and X_μ are path connected, there are paths $\sigma : x \simeq x'$ and $\tau : y' \simeq y$. Furthermore, since X_{λ_0} is path connected, there exists a path $\gamma : x' \simeq y'$. The path product $(\sigma\gamma)\tau : x \simeq z$ is defined and joins x and z. Thus X is path connected. □

6.3.11 **Corollary**. *Let X be a topological space and let $X_\lambda \subset X$ be a family of path connected sets, where $\lambda \in \Lambda$, such that $X = \bigcup X_\lambda$. If $\cap X_\lambda \neq \emptyset$, then X is path connected.* □

6.3.12 Exercise. Let X_n be a sequence of path connected subspaces of X such that for all n, $X_n \cap X_{n+1} \neq \emptyset$. Show that $\bigcup_n X_n$ is path connected.

The path-connectedness property is inherited by products. We have the next.

6.3.13 **Proposition**. *If X_λ, $\lambda \in \Lambda$, is a nonempty family of nonempty spaces, then their product $\prod_{\lambda\in\Lambda} X_\lambda$ is path connected if and only if each of its factors X_λ is path connected.*

Proof: Assume that for every λ, X_λ is path connected, and let x, $y \in \prod_{\lambda\in\Lambda} X_\lambda$ be any two points. If x_λ, $y_\lambda \in X_\lambda$ are their components in the factor X_λ, since this factor space is path connected, there is a path $\sigma_\lambda : I \longrightarrow X_\lambda$ that joins them. By the universal property of the product, there is a (unique) path $\sigma : I \longrightarrow \prod_{\lambda\in\Lambda} X_\lambda$ such that composed with the projection on the factor X_λ one obtains σ_λ. Therefore, $\sigma : x \simeq y$.

Conversely, if $\prod_{\lambda\in\Lambda} X_\lambda$ is path connected, since for each $\lambda \in \Lambda$, X_λ is the image under the corresponding projection, then by Theorem 6.3.8, X_λ is path connected. □

Consequently, any Euclidean space $\mathbb{R}^n$ is path connected. More generally, one has the following.

6.3.14 EXAMPLES.

1. Any Euclidean ball, that is, any ball around some point of $\mathbb{R}^n$ is path connected. If the ball is open, then this holds simply because this ball is homeomorphic to the space $\mathbb{R}^n$. If it is closed, it is quite obvious that any boundary point can be joined by a path to any interior point (just take the linear path, namely the straight line segment between both points).

2. More generally, any convex set $C \subset \mathbb{R}^n$ (see 2.6.16) is path connected. Namely, if $x, y \in C$, then the line segment $[x, y] \subset X$. Hence the linear path $\lambda : I \longrightarrow C$ given by $\lambda(t) = (1-t)x + ty$ is a path that joins x and y in C.

6.4 LOCALLY PATH-CONNECTED SPACES

As in the case of connectedness, path-connectedness has a local version.

6.4.1 DEFINITION. We say that a topological space X is *locally path connected* if each point $x \in X$ has a local basis consisting of path-connected neighborhoods.

6.4.2 EXAMPLE. Any Euclidean space $\mathbb{R}^n$ is locally path connected, since each point has a local basis consisting of balls, that are always path connected (see 6.3.14).

6.4.3 **Proposition**. *Let X be a topological space. The following are equivalent:*

(a) *X is locally path connected.*

(b) *For every open set $A \subset X$, the path-components of A are open.*

(c) *X has a basis consisting of open path-connected sets.*

(d) *If $A \subset X$ is open and $q : A \longrightarrow A'$ is a quotient map that identifies each path-component of A in a point, then A' is discrete.*

Proof:

(a) $\Longrightarrow$ (b) If $A \subset X$ is open and $x \in A$, then by (a) there exists $V \in \mathcal{N}_x$ such that $V \subset A$ and V is path-connected. Therefore, $V \subset c_x(A)$, where $c_x(A)$ denotes the path-component of A where x lies. Hence, this path-component is open.

(b) $\Longrightarrow$ (c) Let $\mathcal{B}$ be the family of path-components of open sets of X. By (b) they build a basis for the topology of X.

(c) $\Longrightarrow$ (a) Let $\mathcal{B}$ be a basis for X, whose elements are path connected. If $x \in X$, then $\mathcal{B}_x = \{V \in \mathcal{B} \mid x \in V\}$ is a local basis of path-connected neighborhoods at the point x. Therefore, X is locally path-connected.

(b) $\Longrightarrow$ (d) Let A be open in X and let $q : A \longrightarrow A'$ be the quotient map that sends each path-component of A to a point. Hence, for $x' \in A'$, $q^{-1}(x')$ is a path-component of A, and thus, by (b) it is open. Therefore, since q is an identification, the singular set $\{x'\}$ is open, and thus A' is discrete.

(d) $\Longrightarrow$ (b) Let A be open in X and let $q : A \longrightarrow A'$ be the quotient map that identifies each path-component of A in a point. By (d), A' is discrete, and so each singular set $\{x'\} \subset A'$ is open. Therefore, each path-component of A, $c_x(A) = q^{-1}(q(x))$, is open. □

6.4.4 **Corollary**. *If X is locally path connected, then for each $x \in X$ the path-component of X, c_x, where x lies, is open and closed in X.*

Proof: By 6.4.3(b), c_x is open. Since $X - c_x = \bigcup_{y \notin c_x} c_y$, this set is also open, so that c_x is also closed. □

Now we can establish a converse result of 6.3.6. Example 6.3.7 shows that not every connected space is path connected. However, the space X of that example is not locally path connected. In fact, it is not locally connected.

6.4.5 **Corollary**. *If X is a connected and locally path-connected space, then it is a path-connected space.*

Proof: If $x \in X$, then the path-component of X in x, c_x, is nonempty, open, and closed. Therefore, since X is connected, $c_x = X$, and thus X is path connected. □

To finish this section, we shall give some examples.

6.4.6 EXAMPLES.

(a) The space X of 6.2.3 is connected, but it is not locally path connected, and it is not either path connected. However, if $C = \{(0,y) \mid y \in [-\frac{3}{2}, -1]\} \cup \{(x, -\frac{3}{2}) \in \mathbb{R}^2 \mid x \in [0,1]\} \cup \{(1,y) \mid y \in [-\frac{3}{2}, 0]\}$, then the space $Y = X \cup C$ is path connected, but it is not locally path connected. This space Y is known as the *Polish circle* (see Figure 6.8).

(b) The *infinite comb*, defined in 6.2.2(c) as the subspace of $\mathbb{R}^2$ given by $P = \{(x,y) \in \mathbb{R}^2 \mid y = 0, \text{ or } x = 0, \frac{1}{n};\ n = 1, 2, \ldots \text{ and } 0 \leq y \leq 1\}$, is path connected, but not locally (path) connected.

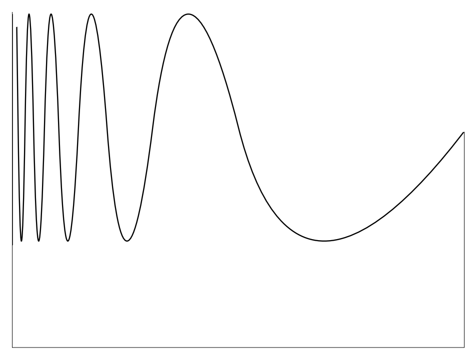


Figure 6.8 The Polish circle

6.4.7 Exercise. Prove that if X is a locally path-connected space, then for every $x \in X$ the connected component C_x coincides with the path-component c_x.

6.4.8 Exercise.

(a) Determine the connected components and the path components of the space $\mathbb{R}^\omega$ as defined in Example 6.1.26.

(b) Show that $x, y \in \mathbb{R}^\omega$ lie in the same component of $\mathbb{R}^\omega$ if and only if the sequence $x - y$ is eventually zero. (*Hint:* Prove that if $x - y$ is not eventually zero, then there is a homeomorphism $\varphi : \mathbb{R}^\omega \longrightarrow \mathbb{R}^\omega$ such that $\varphi(x)$ is bounded and $\varphi(y)$ is unbounded.)

6.4.9 Exercise. Consider in $\mathbb{R}^2$ the set $I_\mathbb{Q}$ of all rational points in the unit interval I and let R be the set $\{1\} \times I_\mathbb{Q}$. Define X as the union of all line segments that join the origin with each of the points of R.

(a) Show that X is path connected.

(b) Show that the only point at which X is locally path connected is the origin.

6.4.10 Exercise. Consider in $\mathbb{R}^2$ the set $I_\mathbb{Q}$ of all rational points in the unit interval I and let X be the union of all line segments $\{q\} \times I$, $q \in I_\mathbb{Q}$ with $I \times \{0\}$.

(a) Show that X is path connected.

(b) At which points is X locally path connected?

6.4.11 Exercise. Consider in $\mathbb{R}^2$ the set $K = \{\frac{1}{n} \mid \mathbb{N}\}$ and let S be the set $\{1\} \times K$. Define X as the union of all line segments that join the origin with each of the points of S.

(a) Show that X is path connected.

(b) At which points is X locally path connected?

6.4.12 Exercise.

(a) Show that the closure of the topologist's sine curve $X = \{0\} \times [-1,1] \cup \{(t, \sin(\frac{\pi}{t})) \mid t \in I\}$ (see 6.2.3(b)) is not path connected. (*Hint:* If $\sigma : I \longrightarrow X$ is a path with start at the origin and end in $\{(t, \sin(\frac{\pi}{t})) \mid t \in I\}$, then the set of those t for which $\sigma(t) \in \{0\} \times [-1,1]$ is closed, thus it has a largest element b. Hence the restriction $\lambda : [b,1] \longrightarrow X$ maps b into $\{0\} \times [-1,1]$ and $(b,1]$ into $\{(t, \sin(\frac{\pi}{t})) \mid t \in I\}$. For each $t \in [b,1]$, put $\lambda(t) = (x(t), y(t))$. Thus $x(b) = 0$ and $x(t) > 0$ and $y(t) = \sin(\frac{\pi}{x(t)})$ for $t > b$. Show that there is a sequence of points $t_k \to b$ such that $y(t_k) = (-1)^k$, thus not convergent. This contradicts the continuity of σ.)

(b) Conclude that the polish circle (see 6.4.6) is not locally path connected.

Chapter 7 Filters

Filters are a concept defined on sets. They constitute a powerful tool to prove fundamental results in topology. Among other things, their good behavior with respect to cartesian products allows to use them to prove theorems which state that products have certain properties if and only if their factors have them. One important example of this fact is the Tychonoff theorem on compactness, which making use of filters we shall prove in the next chapter.

Filters codify and extend in some way the behavior of sequences. In the topological setup one has a concept of convergence of filters. On the other hand, the set of neighborhoods of a point are an example of a filter. In this sense, filters are a common generalization of both sequences and neighborhoods, which is very useful to better understand both concepts. Filters allow to study convergence in arbitrary topological spaces. These properties reduce to properties of sequence-convergence in first-countable spaces, i.e. spaces for which each point has a countable neighborhood basis. Such properties of filters can be inferred from the analysis of sequence convergence.

7.1 Filters

In this section we shall introduce the concepts of a filter basis and of a filter. We shall study the relationships between different filters, as well as of maximal filters which will be called ultrafilters. Making use of filters we shall generalize the concept of sequence convergence and we shall reformulate classical concepts about convergence in metric spaces.

There is a more detailed treatment of filters in other texts such as [15], [20], or [7].

We start recalling some concepts already defined in the case of metric spaces in Section 1.4.

7.1.1 Definition. A *sequence* in a topological space (or in a set) X is a function $\mathbb{N} \longrightarrow X$, $n \mapsto x_n$. This function is denoted by (x_n) or simply by x_n. We say that the sequence *converges* to x, in symbols $x_n \to x$, if for every neighborhood V of x there is a number $N \in \mathbb{N}$ such that $x_n \in V$ if $n \geq N$. In this case, we say that x is a *limit* of the sequence.

For each $N \in \mathbb{N}$ we call *tail* of (x_n) the restriction $(x_n)_{n>N}$.

Hence a sequence converges to a point if for each neighborhood of this point there is a tail of the sequence which is contained in this neighborhood.

We start observing the following.

7.1.2 NOTE. Assume that a space X is first-countable (see 2.4.3), and let $\{U_n\}$ be a countable neighborhhood basis at $x \in X$. We can always assume that for all n, $U_n \supseteq U_{n+1}$ for all n. Such a neighborhood basis is said to be *nested*. If the given basis is not already nested, we may replace it by the neighborhood basis $\{U'_n\}$, where $U'_n = U_1 \cap \cdots \cap U_n$.

The next is an easy exercise.

7.1.3 EXERCISE. Assume that $\{U_n\}$ is a nested countable neighborhood basis at $x \in X$, and for each n take any point $x_n \in U_n$ Show that $x_n \to x$.

The following result characterizes the topology of a first-countable space, as well as the continuity of a map, whose domain is first-countable, using sequence convergence.

7.1.4 **Theorem**. *Let X be a first-countable topological space. Then the following hold:*

(a) *$A \subset X$ is closed if and only if for every sequence (x_n) in A such that $x_n \to x$ in X, one has that $x \in A$.*

(b) *$f : X \longrightarrow Y$ is continuous at $x \in X$ if and only if for every sequence (x_n) in X such that $x_n \to x$, the image sequence $f(x_n) \to f(x)$.*

Proof: (a) Assume first that $A \subset X$ is closed. Let (x_n) be a sequence of elements of A such that $x_n \to x$ in X. If $x \notin A$, then $X - A$ is an open neighborhood of x and thus there is a natural number N such that $x_n \in X - A$ for all $n \geq N$. This is a contradiction, since $x_n \in A$ for all n.

Conversely, take an arbitrary point $x \in \overline{A}$. Since X is first-countable, there is a nested neighborhood basis of x, say $\{U_n\}$. Since $U_n \cap A \neq \emptyset$, we may take a point

$x_n \in A \cap U_n$, and since the neighborhood base is nested, it follows by Exercise 7.1.3 that $x_n \to x$. Therefore, by assumption, $x \in A$. Consequently $\overline{A} \subset A$ and therefore A is closed.

(b) Assume first that f is continuous at x and suppose $x_n \to x$. Take a neighborhood V of $f(x)$. Since f is continuous at x, there is a neighborhood U of x such that $f(U) \subseteq V$. Furthermore, since $x_n \to x$, there is an N such that if $n \geq N$, then $x_n \in U$. Therefore, if $n \geq N$, then $f(x_n) \in f(U) \subseteq V$ and thus, $f(x_n) \to f(x)$.

Conversely we shall prove that under the assumption, for all $A \subseteq X$, $f(\overline{A}) \subset \overline{f(A)}$. Thus by Theorem 2.5.8 (d), f shall be continuous. So take an arbitrary point $x \in \overline{A}$. Since X is first-countable, there is a nested neighborhood basis of x, say $\{U_n\}$. On the other hand, $U_n \cap A \neq \emptyset$, so we may take a point $x_n \in A \cap U_n$. Since the neighborhood base is nested, once again it follows by Exercise 7.1.3 that $x_n \to x$. Hence, by assumption $f(x_n) \to f(x)$, and since $f(x_n) \in f(A)$ for all n, $f(x) \in \overline{f(A)}$. □

7.1.5 NOTE. Observe that in the first part of the proofs of (a) and (b) in the previous theorem, the assumption that X is first countable is not necessary. But for the second parts the assumption is needed as the following exercise shows.

7.1.6 EXERCISE. Consider an uncountable set X with the *cocountable* topology (2.1.2(f)) on it, namely, $A \subset X$ is open if and only if either $A = \emptyset$ or $X - A$ is at most countable. Show that a sequence in X is convergent if and only if the sequence is finally constant, i.e. it has a constant tail. Hence any function $f : X \longrightarrow Y$ will map convergent sequences in convergent sequences. However not any function f must be continuous. Give examples of Y and f such that f is not continuous. Show explicitly that X is not first-countable.

In general, as we saw above, without the first-countability assumption, Theorem 7.1.4 is not true. Therefore, we must generalize the concept of sequence convergence. To do that, consider the following analogy between sequences and neighborhoods of a point: The intersection of two tails of a sequence is again a tail. Analogously, the intersection of two neighborhoods of a point is again a neighborhood of that point. This analogy can be extended to basic neighborhoods, observing that the intersection of two of them contains another. These observations suggest the next definition.

7.1.7 DEFINITION. A nonempty system $\mathcal{B}$ of nonempty subsets of a set X is called *filter basis* in X if the following holds:

(FB) $B_1, B_2 \in \mathcal{B} \Longrightarrow \exists B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$.

7.1.8 EXAMPLES.

(a) If $A \subset X$ is nonempty, then the set $\{A\}$ is a filter basis.

(b) Let (x_n) be a sequence in the set X. Then the system $\mathcal{B}$ of tails of (x_n) is a filter basis in X.

(c) Let X be topological space and take $x \in X$. A system $\mathcal{B}$ of basic neighborhoods in x is a filter basis in X.

(d) Let X_λ, $\lambda \in \Lambda$, be a nonempty family of nonempty sets. If $\mathcal{B}_\lambda$ is a filter basis in X_λ, then the products

$$B_{l_1} \times \cdots \times B_{l_k} \times \prod_{\lambda \neq l_1 \dots l_k} X_\lambda \,,$$

such that $B_{l_j} \in \mathcal{B}_{l_j}$, $j = 1, \dots, k$, constitute a filter basis in $\prod X_\lambda$.

7.1.9 DEFINITION. A nonempty system $\mathcal{F}$ of nonempty subsets of a set X is *filter* in X if

(F1) $F \in \mathcal{F}$, $F' \supset F \Longrightarrow F' \in \mathcal{F}$.

(F2) $F_1, F_2 \in \mathcal{F} \Longrightarrow F_1 \cap F_2 \in \mathcal{F}$.

Notice that necessarily $X \in \mathcal{F}$ for every filter $\mathcal{F}$.

7.1.10 EXAMPLES.

(a) Given a filter basis $\mathcal{B}$ in a set X, the family of all supersets of the sets in $\mathcal{B}$ constitutes a filter $\mathcal{F}$, called the *filter generated by* $\mathcal{B}$. Namely

$$\mathcal{F} = \{F \subseteq X \mid \exists B \in \mathcal{B} \text{ such that } B \subseteq F\}\,.$$

We shall say that $\mathcal{B}$ is a *basis* of $\mathcal{F}$.

(b) Let $X \neq \emptyset$, then $\{X\}$ is a filter in X called the *trivial filter*. It is the filter generated by the filter basis $\{X\}$.

(c) All supersets of nonempty set $A \subset X$ build a filter denoted by $\mathcal{F}_A$, namely $\mathcal{F}_A = \{F \subseteq X \mid A \subseteq F\}$. It is the filter generated by the filter basis $\{A\}$.

(d) Given a sequence (x_n) in a set X, the supersets of the tails of the sequence form a filter called *elementary filter* generated by the sequence. It is denoted by $\mathcal{F}_{(x_n)}$. Namely

$$\mathcal{F}_{(x_n)} = \{F \subseteq X \mid \exists N \in \mathbb{N} \text{ such that } x_n \in F \text{ for all } n > N\}\,.$$

It is the filter generated by the filter basis of tails of the sequence.

(e) Let X be a topological space and take $x \in X$. The set $\mathcal{N}_x$ of all neighborhoods of x is a filter in X called *neighborhood filter* of $x \in X$. This is always a filter generated by a neighborhood basis, seen as a filter basis.

(f) Let X be an infinite set. The system of *cofinite* sets of X, namely sets whose complement is finite, is a filter in X called *cofinite filter.*

(g) Every filter is a filter basis.

7.1.11 Definition. We shall say that two filter bases are *equivalent* if both generate the same filter according to Example 7.1.10(a).

7.1.12 Example. Consider $\mathcal{B} = \{(-1/n, 1/n) \subset \mathbb{R} \mid n \in \mathbb{N}\}$ and take $\mathcal{B}' = \{(-\pi/2^n, \pi/2^n) \subset \mathbb{R} \mid n \in \mathbb{N}\}$. Both sets are filter bases and they are equivalent since they generate precisely the neighborhood filter of 0 in $\mathbb{R}$.

As we saw in Example 7.1.10(d), every sequence determines an elementary filter. Moreover different sequences may determine the same filter, for instance sequences only differ in a finite number of their terms.

If we observe the definition of convergence of a sequence in a topological space, we see this convergence is based in a comparison of the tails of the sequence and the neighborhoods of the point to which the sequence converges. More precisely, the sequence (x_n) converges to a point x if for each neighborhood of x there is a tail of the sequence contained in the neighborhood. In other words, the sequence (x_n) converges to x if each neighborhood of x is an element of the elementary filter generated by the sequence. Or in symbols, $x_n \to x$ if $\mathcal{N}_x \subset \mathcal{F}_{(x_n)}$. This suggests the next concept.

7.1.13 Definition. Let $\mathcal{F}$ and $\mathcal{G}$ be two filters in a set X. If $\mathcal{F} \subset \mathcal{G}$, then we say that $\mathcal{F}$ is *coarser* than $\mathcal{G}$ and that $\mathcal{G}$ is *finer* than $\mathcal{F}$.

7.1.14 Examples.

(a) The trivial filter $\{X\}$ is the coarsest filter in X.

(b) There is no finest filter of all. Namely, if A and $X - A$ are nonempty, then each of them determines a superset filter. Namely the filters $\mathcal{F}_A$ y $\mathcal{F}_{X-A}$. There can be no filter finer than both, then otherwise this filter would have A and $X - A$ as elements, and hence also their intersection $A \cap (X - A) = \emptyset$, which is impossible.

7.1.15 **Theorem**. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be filter bases in the set X. The following are equivalent:*

(a) *The filter generated by $\mathcal{B}_1$ is finer than the filter generated by $\mathcal{B}_2$.*

(b) *Each set in $\mathcal{B}_2$ contains one set in $\mathcal{B}_1$.* □

7.1.16 **Corollary**. *Two filter bases $\mathcal{B}_1$ and $\mathcal{B}_2$ in a set X are equivalent if and only if each set in $\mathcal{B}_1$ contains one set in $\mathcal{B}_2$ and viceversa.* □

Notice that Example 7.1.12 illustrates quite well the statement of the previous corollary. From the considerations previous to Definition 7.1.13, the following concept is quite natural.

7.1.17 Definition. Let X be a topological space and let $\mathcal{F}$ be a filter in X. We say that the filter $\mathcal{F}$ *converges* to a point $x \in X$, in symbols $\mathcal{F} \to x$, if the filter $\mathcal{F}$ is finer than the neighborhood filter of x. I.e. $\mathcal{F} \to x$ if $\mathcal{N}_x \subset \mathcal{F}$. If $\mathcal{B}$ is a filter basis in X, then we say that $\mathcal{B}$ *converges* to $x \in X$, in symbols $\mathcal{B} \to x$, if the filter generated by $\mathcal{B}$ converges to x. We write $\lim \mathcal{F}$ for the set of all points $x \in X$ such that $\mathcal{F}$ converges to x, i.e. in symbols $\lim \mathcal{F} = \{x \in X \mid \mathcal{F} \to x\}$. We call this the *limit of*$\mathcal{F}$. If $\lim \mathcal{F}$ has only one point x, then we write $\lim \mathcal{F} = x$. If $\lim \mathcal{F} = \emptyset$ then we say that $\mathcal{F}$ *does not converge* or simply that it *diverges*.

7.1.18 **Proposition**. *Let a filter $\mathcal{G}$ be finer than a filter $\mathcal{F}$. If $\mathcal{F}$ converges to x, then $\mathcal{G}$ converges to x too.* □

7.1.19 Examples.

(a) Let X be a topological space and take a sequence (x_n) in X. Then (x_n) converges to x if and only if the elementary filter generated by (x_n) converges to x.

(b) Let X be an indiscrete space. Then for every filter $\mathcal{F}$ and for every point x in X, $\mathcal{F} \to x$, i.e. $\lim \mathcal{F} = X$.

7.1.20 **Proposition**. *Let $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of filters in a set X. Then the intersection $\mathcal{F} = \bigcap \mathcal{F}_\lambda$ is a filter.* We call this filter simply the *intersection* of the filters $\mathcal{F}_\lambda$, or *infimum* of $\{\mathcal{F}_\lambda\}$. We denote it by $\inf \mathcal{F}_\lambda$.

If $\mathcal{G}$ is a filter in X which is coarser than $\mathcal{F}_\lambda$ for all $\lambda \in \Lambda$, then $\mathcal{G} \subset \mathcal{F}_\lambda$ for all $\lambda \in \Lambda$. Clearly $\mathcal{G}$ is also coarser than $\inf \mathcal{F}_\lambda$, namely, $\inf \mathcal{F}_\lambda$ is the finest of all filters that are coarser than all $\mathcal{F}_\lambda$.

7.1.21 **Theorem**. *Let X be a topological space. The infimum of all filters which converge to $x \in X$ is the neighborhood filter $\mathcal{N}_x$.* □

Let now $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ be a family of filter bases in a set X and let $\mathcal{F}_\lambda$ be the filter generated by $\mathcal{B}_\lambda$. If there is a filter which is finer than all filters $\mathcal{F}_\lambda$, then it must contain all sets of all the $\mathcal{F}_\lambda$, or of all $\mathcal{B}_\lambda$. It must also contain all finite intersections of sets in the filter bases $\mathcal{B}_\lambda$. This suggests to define

$$\mathcal{B} = \{\bigcap_{i=1}^{n} B_i \mid B_i \in \mathcal{B}_{\lambda_i}, \lambda_i \in \Lambda\}\,,$$

and we clearly have the next.

7.1.22 **Proposition**. *If no set in $\mathcal{B}$ is empty, then $\mathcal{B}$ is a filter basis.* □

Under the assumption of 7.1.22, $\mathcal{B}$ is a basis of the coarsest filter which is finer than all filters $\mathcal{F}_\lambda$. Thus we have the following result.

7.1.23 **Theorem**. *Let $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ be a family of filter bases in a set X. Then there is a finer filter than all filters generated by the filter bases $\mathcal{B}_\lambda$ if and only if the finite intersections $\bigcap_{i=1}^n B_i$, where $B_i \in \mathcal{B}_{\lambda_i}$, $\lambda_i \in \Lambda$ (different), are nonempty. If this is the case, then these intersections constitute a basis of the coarsest filter which is finer than all filters generated by the filter bases $\mathcal{B}_\lambda$.* □

7.1.24 Definition. Let $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ be a family of filter bases in a set X. Then the filter $\mathcal{F}$ generated by the filter basis $\mathcal{B} = \{\bigcap_{i=1}^n B_i \mid B_i \in \mathcal{B}_{\lambda_i}, \lambda_i \in \Lambda\}$ is called *supremum* of the filters $\mathcal{F}_\lambda$ generated by the $\mathcal{B}_\lambda$. In symbols, $\mathcal{F} = \sup \mathcal{F}_\lambda$.

7.1.25 **Theorem**. *Let X be a set and take $A \subset X$. Let $\mathcal{F}$ be a filter in X. Then $A \in \mathcal{F}'$ for some filter $\mathcal{F}'$ finer than $\mathcal{F}$ if and only if $X - A \notin \mathcal{F}$.*

Proof: If $X - A \in \mathcal{F}$, then $X - A \in \mathcal{F}'$ for all $\mathcal{F}'$ finer than $\mathcal{F}$. Hence $A \notin \mathcal{F}'$ for all $\mathcal{F}'$ finer than $\mathcal{F}$.

Conversely, if $X - A \notin \mathcal{F}$ and $F \in \mathcal{F}$, then $F \not\subset X - A$. Hence $F \cap A \neq \emptyset$. By the previous theorem, the supremum of $\mathcal{F}$ and the filter $\mathcal{F}_A$ of all supersets of A must exist, i.e. there would exist a filter $\mathcal{F}'$, which is finer than $\mathcal{F}$ and has A as an element. □

7.1.26 Exercise. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be filter bases. Show that $\mathcal{B} = \{B_1 \cup B_2 \mid B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2\}$ is a filter basis. What relationship keeps the filter generated by $\mathcal{B}$ with the filters generated by $\mathcal{B}_1$ y $\mathcal{B}_2$, respectively?

We now consider the following question. Which property must X have in order that $\lim \mathcal{F} \subseteq \{x\}$, $x \in X$? That is, under what condition on X does a filter have at most one limit point. This requires that there is no filter finer than the neighborhood filters of any two points of X. We shall study this condition in detail in what follows.

Let X be a topological space, take points $x \neq y$ in X. There is a finer filter than the neighborhood filter of x, $\mathcal{N}_x$, and the neighborhood filter of y, $\mathcal{N}_y$, if and only if every neighborhood of x meets all neighborhoods of y. In other words, there is a filter $\mathcal{F}$ such that $\mathcal{F} \to x$ and $\mathcal{F} \to y$ if and only if there is a filter $\mathcal{F}$ such that $\mathcal{N}_x \subset \mathcal{F}$ and $\mathcal{N}_y \subset \mathcal{F}$. This is equivalent, as we said above, to the fact that each neighborhood of x intersects each neighborhood of y.

Thus we have proved the following.

7.1.27 **Proposition**. *Any filter in X converges to at most one point if and only if the following condition holds:*

(H) *For each pair of points $x \neq y$ in X, there are neighborhoods $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ such that $U \cap V = \emptyset$.* □

Condition (H) is known as *Hausdorff separability axiom.*

7.1.28 **Proposition**. *Axiom* (H) *is equivalent to the following:*

(H′) *Each point of X is the intersection of all its closed neighborhoods.*

Proof:

(H)$\Longrightarrow$(H′) Let $x \in X$ be an arbitrary point of X. For each point $y \neq x$, there are open disjoint neighborhoods U and V of x and y, respectively. Hence

$W = X - V \supset U$, so that W is a closed neighborhood of x that does not contain y as an element. Therefore $\{x\} = \bigcap\{W \mid W \text{ is a closed neighborhood of } x\}$.

(H′)$\Longrightarrow$(H) Take $x, y \in X$, $x \neq y$. There is a closed neighborhood V of y such that $x \notin V$. Hence $X - V$ is an open set that contains x as an element. Thus $U = X - V$ is a neighborhood of x which does not intersect V. □

7.1.29 Definition. The set $\Delta = \{(x, x) \in X \times X\}$ is called the *diagonal* of the space X in $X \times X$.

Clearly $(x, y) \notin \Delta$ if and only if $x \neq y$. If X satisfies (H) and $(x, y) \notin \Delta$, then there are disjoint neighborhoods U and V of x and y, respectively. The product $U \times V$ is a neighborhood of (x, y) in $X \times X$ such that $(U \times V) \cap \Delta = \emptyset$, since if a diagonal point $(z, z) \in U \times V$, then $z \in U \cap V$. Hence Δ is closed in $X \times X$.

Conversely, if Δ is closed in $X \times X$, then the difference set $X \times X - \Delta$ is open and if $x \neq y$, i.e. if $(x, y) \in X \times X - \Delta$, then there is a neighborhood W of (x, y) in $X \times X$, which does not meet Δ. Hence there are neighborhoods U and V of x y y in X, respectively, such that $U \times V \subset W$. This way $(U \times V) \cap \Delta = \emptyset$ or equivalently $U \cap V = \emptyset$.

Thus we have shown the following result.

7.1.30 **Proposition**. *Axiom* (H) *is equivalent to the following:*

(H″) *The diagonal* $\Delta \subset X \times X$ *is closed.* □

The two previous propositions can be summarized in the next result.

7.1.31 **Theorem**. *Let X be a topological space. The following are equivalent:*

(H) *Any two different points of X have disjoint neighborhoods.*

(H′) *Any point of X is the intersection of all its closed neighborhoods.*

(H″) *The diagonal in $X \times X$ is closed.*

(H‴) *No filter in X converges to more than one point.* □

7.1.32 Definition. A topological space X that satisfies one (and thus all) of the axioms (H)–(H‴), is called *Hausdorff space* or T_2 *space*.

7.1.33 Examples.

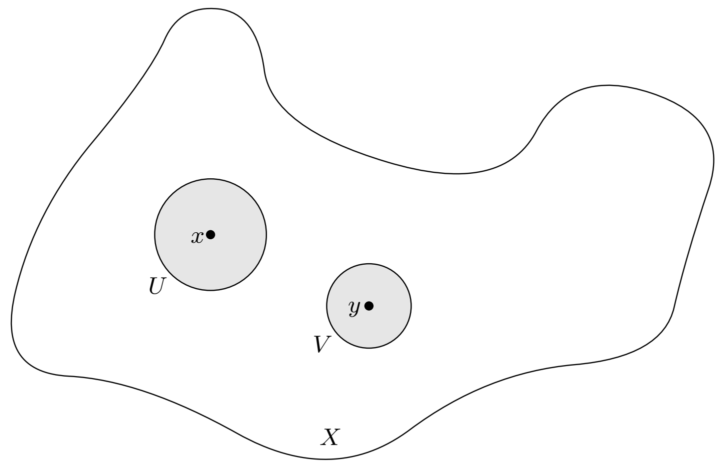


Figure 7.1 Disjoint neighborhoods of different points

(a) Every metric space X is Hausdorff. Namely, given $x \neq y$ in X and $\varepsilon = d(x,y)/2$, then $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$.

(b) Not every pseudometric space is Hausdorff. For instance the indiscrete space with more than one element is not Hausdorff..

The property of being a Hausdorff space is *hereditary*, i.e. is a property that is inherited by all subspaces.

7.1.34 **Proposition**. *Every subspace of a Hausdorff space is a Hausdorff space, i.e. the Hausdorff separability property is hereditary.* □

Since in a Hausdorff space each point is the intersections of all its closed neighborhoods, we have the following.

7.1.35 **Proposition**. *In a Hausdorff space every one-point set is closed.* □

The converse of this proposition is false, as the next example shows.

7.1.36 EXAMPLE. Let X be an infinite set with the cofinite topology (see 2.1.2(e)). It is obvious that each one-point set is closed. However, the space is not Hausdorff.

7.1.37 **Proposition**. *A topological space X is such that each of its one-point sets is closed if and only if it satisfies*

(T$_1$) For every pair of points $x \neq y$ in X there are neighborhoods U and V of x and y in X, respectively, such that $y \notin U$ and $x \notin V$. □

7.1.38 DEFINITION. Axiom (T$_1$) is known as the *first separability axiom.* A topological space X which satisfies the axiom (T$_1$) is called a T$_1$-*space.* (some authors call these *Fréchet spaces*, but this might lead to confusion and we avoid this name).

7.1.39 EXERCISE. Let X be an infinite set. Show that the coarsest topology on X such that it becomes a T$_1$-space is the cofinite topology.

7.1.40 EXERCISE. Show that $\mathbb{R}^n$ with the Zariski topology (2.2.23) is T$_1$-space, which is not Hausdorff. What relation has this example with 7.1.36?

7.1.41 NOTE. Axiom (H) is sometimes denoted by (T$_2$) and is also is known as the *second separability axiom.* A topological space X which satisfies axiom (T$_2$), i.e. a Hausdorff space, is also called a T$_2$-*space.* Axiom (T$_1$) is weaker than (T$_2$), namely (T$_2$) $\Longrightarrow$ (T$_1$).

There is a weaker axiom than (T$_1$), namely

(T$_0$) For every pair of points $x \neq y$ in X there is either a neighborhood U of x in X, or a neighborhood V of y in X, such that $y \notin U$ or $x \notin V$. Axiom (T$_0$) is known as the *zeroth separability axiom.*

We have

$$(\mathrm{T}_2) \Longrightarrow (\mathrm{T}_1) \Longrightarrow (\mathrm{T}_0).$$

7.1.42 EXERCISE. Show in an example that not all T$_0$-space is a T$_1$-space. (*Hint:* Consider the Sierpinski space.)

Exercises 7.1.42 and 7.1.36 or 7.1.40 show that no two of the three separability axioms are equivalent.

7.1.43 EXERCISE. Let X be a metric space with metric d and let (x_n) be a sequence in X. Show that $x_n \to x$ if and only if $d(x_n, x) \to 0$.

To finish this section we shall see how a filter in a set generates a topology on it.

7.1.44 Example. Let X be a set and $\mathcal{F}$ an arbitrary filter in X. Take the set $X^* = X \cup \{\infty\}$ and for $x \in X \subset X^*$ define its neighborhood filter by $\mathcal{N}_x = \mathcal{F}_x$, namely the filter of all sets in X^* that contain $\{x\}$. For $\infty \in X^*$, define the neighborhood filter by $\mathcal{N}_\infty = \{F \cup \{\infty\} \mid F \in \mathcal{F}\}$. It is an *exercise* to show that these sets make up a neighborhood system on the set X^*, and that they determine a topology on X^*. Denote the resulting topological space by $X^*_{\mathcal{F}}$.

It is an interesting *exercise* to consider the special case in which X is an infinite set and $\mathcal{F}$ is the cofinite filter described in 7.1.10(f).

7.1.45 Exercise. Show that the subspace $X \subset X^*_{\mathcal{F}}$ (with the relative topology) is discrete and is not closed. Conclude that $\overline{X} = X^*_{\mathcal{F}}$, namely, X is dense in $X^*_{\mathcal{F}}$ (see Definition 7.4.17).

7.2 Cluster points

Following the parallelism between sequences and filters, the concept of a cluster point of a sequence can be extended to filters. In this section we shall study the concept of cluster point of a filter and what relation it has with comparable filters. We start with the special case of cluster point of a sequence.

7.2.1 Definition. Let X be a topological space and $\{x_n\}$ a sequence of points in X. A point $x \in X$ is *cluster point* de (x_n) if for all $U \in \mathcal{N}_x$ $x_n \in U$ for infinitely many values of n. Equivalently x is a cluster point of (x_n) if and only if for all $U \in \mathcal{N}_x$, U meets every tail of the sequence (x_n) or, equivalently, x is a limit point of each tail of the sequence.

As we already said, the concept of sequence appears to be poor if we consider it in very general topological spaces. The following example due to R. Arens shows the pathological behavior of sequences and justifies the use of of filters as alternative to study convergence.

7.2.2 Example. Let $X = (\mathbb{N} \times \mathbb{N}) \cup \{(0,0)\}$ have the topology such that $\mathbb{N} \times \mathbb{N}$ is discrete and the neighborhoods of $(0,0)$ are the sets $U \subset X$ for which $(0,0) \in U$ and there is an $N \in \mathbb{N}$ such that for all $n \geq N$, U contains all points of $\{n\} \times \mathbb{N}$ but finitely many. This topology is Hausdorff.

The diagonal sequence, namely, the sequence given by $x_n = (n,n)$ has $(0,0)$ as a cluster point. However it does not contain any subsequence that converges to $(0,0)$.

7.2.3 Exercise. Check all statements in the previous example.

7.2.4 Exercise. Let X be a first-countable space and let (x_n) be a sequence in X with a cluster point x_0. Show that (x_n) has a subsequence that converges to x_0.

Starting from the idea that the tails of a sequence form a filter basis, we formulate the following analogy. Let $\mathcal{F}$ be a filter in X and let $\mathcal{B}$ be a filter basis of $\mathcal{F}$. If x is a limit point of every set in $\mathcal{B}$, then it is a limit point of every set in $\mathcal{F}$. In other words, $x \in \overline{A}$ foe every $A \in \mathcal{B}$. If we take $B \in \mathcal{F}$, then there is an $A \in \mathcal{B}$ such that $A \subset B$. Hence $x \in \overline{B}$ for every $B \in \mathcal{F}$. Therefore $x \in \bigcap\{B \in \mathcal{F} \mid B$ is closed$\}$. Conversely, it is clear that if $x \in \bigcap\{B \in \mathcal{F} \mid B$ is closed$\}$, then $x \in \overline{A}$ for every $A \in \mathcal{B}$. Thus we have the next definition.

7.2.5 Definition. Let $\mathcal{F}$ be a filter in a topological space X. We say that $x \in X$ is a *cluster point* of $\mathcal{F}$ if for every $U \in \mathcal{N}_x$ and for all $F \in \mathcal{F}$, $U \cap F \neq \emptyset$.

We have the following assertion.

7.2.6 **Proposition**. *A point x is a cluster point of a filter $\mathcal{F}$ in a space X if and only if $x \in \bigcap\{B \in \mathcal{F} \mid B$ is closed$\}$.* □

From the previous proposition, it is clear that the definition of a cluster point of a filter is consistent with that of a cluster point of a sequence (7.2.1). We have the following.

7.2.7 **Corollary**. *A point x in X is a cluster point of a sequence (x_n) if and only if x is a cluster point of the elementary filter $\mathcal{F}_{(x_n)}$ defined by the sequence.* □

More generally we have the next result.

7.2.8 **Theorem**. *Let X be a topological space and let $\mathcal{F}$ be a filter in X. Then x is a cluster point of $\mathcal{F}$ if and only if x is a cluster point of $\mathcal{F}'$ for every filter $\mathcal{F}'$ coarser than $\mathcal{F}$.* □

The relation of the cluster points of a filter with a filter basis that generates the filter is established in the next result, whose proof is immediate.

7.2.9 **Theorem**. *Let X be a topological space and let $\mathcal{B}$ be a filter basis in X. Then the following are equivalent:*

(a) *x is a cluster point of the filter $\mathcal{F}$ generated by $\mathcal{B}$.*

(b) *For each $U \in \mathcal{N}_x$ and for each $B \in \mathcal{B}$, $U \cap B \neq \emptyset$.*

(c) *$x \in \overline{A}$ for all $A \in \mathcal{B}$.* □

We have, in particular, that if A is a nonempty subset of a topological space X, then $\{A\} = \mathcal{B}$ is a filter basis for the filter $\mathcal{F}_A$ of all supersets of A. Thus analogously to 7.2.7, we get the next.

7.2.10 **Proposition**. *Let X be a topological space and take $A \subset X$. Then $x \in \overline{A}$, i.e. x is a cluster point of A, if and only if x is a cluster point of the filter $\mathcal{F}_A$ of all supersets of A. Hence $\overline{A} = \{x \mid x \text{ is a cluster point of } \mathcal{F}_A\}$.* □

This last result shows also the consistency of the concept of a cluster point of a filter with that of a set.

Theorem 7.2.9 states that the fact that x is a cluster point of the filter $\mathcal{F}$ generated by $\mathcal{B}$ and the fact that every neighborhood of x meets every set in $\mathcal{F}$ are equivalent. Thus the statement is also equivalent to saying that there is a finer filter than both $\mathcal{F}$ and $\mathcal{N}_x$. In other words, that there is a finer filter than $\mathcal{F}$ which converges to x. Thus we have the following result.

7.2.11 **Theorem**. *Let X be a topological space and let $\mathcal{F}$ be a filter in X. Then x is a cluster point $\mathcal{F}$ if and only if there is a filter $\mathcal{G}$ finer than $\mathcal{F}$ such that $\mathcal{G}$ converges to x. In particular, if $\mathcal{F}$ converges to x, then x is a cluster point of $\mathcal{F}$.*□

If a topological space X is not Hausdorff, then a filter might have a limit and several cluster points.

7.2.12 EXAMPLE. Let X be the Sierpinski space, namely $X = \{x, y\}$ with the topology $\mathcal{A} = \{X, \emptyset, \{x\}\}$. The filter $\mathcal{F} = \{X\}$ converges to y, however x is a cluster point of $\mathcal{F}$, although $\mathcal{F}$ does not converge to x.

We have the following result.

7.2.13 **Theorem**. *If X is a Hausdorff space, then for each convergent filter $\mathcal{F}$, the only point in $\lim \mathcal{F}$ is the only cluster point of $\mathcal{F}$.*

Proof: Let $x = \lim \mathcal{F}$ and let y be a cluster point of $\mathcal{F}$. Hence there is a filter $\mathcal{G}$ finer than both $\mathcal{F}$ and $\mathcal{N}_y$. Thus $\mathcal{G} \to y$. But clearly $\mathcal{G} \to x$ too. Since X is Hausdorff, $x = y$. □

7.2.14 Exercise. Let $\mathcal{B} = \{(a, \infty) \subset \mathbb{R} \mid a \in \mathbb{R}\}$.

(a) Show that $\mathcal{B}$ is a filter basis.

The filter $\mathcal{F}$ generated by $\mathcal{B}$ is called *Fréchet filter* in $\mathbb{R}$.

(b) Show that the Fréchet filter in $\mathbb{R}$ with the usual topology does not have cluster points.

(c) Let L be the wrapped line (véase 2.4.7). Show that the Fréchet filter in L converges to 0.

Similarly to sequences, a filter might have no cluster point or just one cluster point without being convergent.

7.2.15 Exercise. Give the details of the proof of 7.2.7, namely, of the fact that x is a cluster point of a sequence (x_n) if and only if x is a cluster point of the elementary filter $\mathcal{F}_{(x_n)}$ associated to $\{x_n\}$. Analogously prove that $x = \lim x_n$ if and only if $x = \lim \mathcal{F}_{(x_n)}$.

Take a first-countable space X. In terms of filters, this means that for all $x \in X$, the neighborhood filter $\mathcal{N}_x$ admits a countable basis. We may consider any filters which admit a countable basis.

7.2.16 Exercise. If $\mathcal{F}$ has a countable basis, then then it has a *nested* basis $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$, namely a filter basis such that if $m > n$, then $B_m \subset B_n$.

7.2.17 **Theorem**. *If $\mathcal{F}$ admits a countable basis, then there is an elementary filter which is finer than $\mathcal{F}$. Moreover, $\mathcal{F}$ is the intersection of all elementary filters that are finer than $\mathcal{F}$.*

Proof: Let $\mathcal{B} = \{B_n\}$ be a nested countable basis of $\mathcal{F}$ (as in 7.2.16) and take $x_n \in B_n$. It is clear that the elementary filter $\mathcal{F}_{(x_n)}$ is finer que $\mathcal{F}$.

Conversely, take $\mathcal{G} = \bigcap\{\mathcal{F}_{(x_n)} \mid \mathcal{F}_{(x_n)} \supset \mathcal{F}\}$. Clearly, $\mathcal{G}$ is finer than $\mathcal{F}$. If $\mathcal{G} \neq \mathcal{F}$, then there is a set $A \in \mathcal{G}$ such that $A \notin \mathcal{F}$. Thus $A \not\supset B_n$ for all $n \in \mathbb{N}$, i.e. $B_n \cap (X - A) \neq \emptyset$. Take any $x_n \in B_n \cap (X - A)$. Obviously $\mathcal{F}_{(x_n)}$ is finer than $\mathcal{F}$ However $A \notin \mathcal{F}_{(x_n)}$, since $\{x_n\} \subset X - A$. Thus $X - A \in \mathcal{F}_{(x_n)}$. This is a contradiction, hence $\mathcal{G} = \mathcal{F}$. □

The theorem above states, in particular, that if a point $x \in X$ has a countable neighborhood basis, then its neighborhood filter $\mathcal{N}_x$ is determined by all sequences in X that converge to x and, by definition of filter convergence, all filters that converge to x are also determined by the same sequences.

However one should notice and take care that among all filters converging to x, filters may exist which do not admit a countable basis. Namely, if $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{G}$ admits a countable basis, this does not mean that $\mathcal{F}$ admits countable basis.

7.2.18 EXAMPLE. The filter $\mathcal{F}_{\{x\}}$ of all supersets of x is finer than its neighborhood filter $\mathbb{N}_x$. A base for $\mathcal{F}_{\{x\}}$ is $\{x\}$, which is (at most) countable). However, if X is not first-countable, the filter $\mathbb{N}_x$ might not admit a countable basis. (It is also possible that if a filter does not admit a countable basis, there could be a coarser filter which does admit it.)

7.2.19 **Theorem**. *Let X be a first countable space. Let $\mathcal{F}$ be a filter in X which admits a countable basis and let x be a cluster point of $\mathcal{F}$, then there is an elementary filter finer than $\mathcal{F}$ which converges to x.*

Proof: Let $\mathcal{B} = \{B_n\}$ be a basis of $\mathcal{F}$ and let $\mathcal{U} = \{U_n\}$ be a basis of $\mathcal{N}_x$. We may assume that these bases are nested (7.2.16).

If x is a cluster point of $\mathcal{F}$, then $B_n \cap U_n \neq \emptyset$ for all n. Take $x_n \in B_n \cap U_n$. Then clearly the elementary filter $\mathcal{F}_{(x_n)}$ satisfies the statement of the theorem. □

Notice that, in particular, if $\mathcal{F}_A$ is the filter of all supersets of $A \neq \emptyset$ in a first-countable space X, then $x \in \overline{A}$ if and only if there is a sequence (x_n) in A such that $x_n \to x$. Thus we have that for first-countable spaces, sequence-convergence allows us to characterize their topology, see 7.1.4.

7.2.20 EXERCISE. Let X be an uncountable set and take a sequence (x_n) in X, whose points are all different. Show that the filter consisting of all cofinite sets in X (i.e. whose complement is finite) is strictly coarser than the elementary filter $\mathcal{F}_{(x_n)}$ and does not allow a countable basis.

7.2.21 EXERCISE. Let X be a set.

(a) Show that the intersection of two elementary filters in X is an elementary filter.

(b) Show that the intersection of two filters in X, which admit a countable basis, admits a countable basis too.

(c) Show that if the supremum of a countable family of filters exists, then this supremum admits a countable basis.

7.2.22 Exercise. Let X be a topological space and let $\mathcal{B}$ be the filter basis consisting of one nonempty set $A \subset X$. Show that if $A = \{x_0\}$, then $\mathcal{B} \to x_0$.

7.2.23 Exercise. Let X be a topological space and let $\mathcal{F}$ be a filter in X. Show that the set of cluster points of $\mathcal{F}$ is closed (possibly empty).

7.2.24 Exercise. Let X be a Hausdorff space and let $\mathcal{F}$ be a filter in X such that $\mathcal{F} \to x_0$. If $\mathcal{G}$ is finer than $\mathcal{F}$ and x is a cluster point of $\mathcal{G}$, then show that $x = x_0$ and $\mathcal{G} \to x_0$.

7.3 Ultrafilters

In this section we shall study the class of ultrafilters, which are the maximal elements in the class of all filters in a set, with respect to the order relation introduced in Section 7.1. They are filters filters which have particularly interesting properties.

We start with a general definition.

7.3.1 Definition. Let M be a set. We say that M is a *partially ordered set*, also briefly called *poset*, if it has an *order relation* or a *partial order* $\leq$ which fulfills for any $a, b, c \in M$ the axioms (OR1), (OR2), and (OR3) of Definition 3.4.2.

7.3.2 Examples. The following are examples of partially ordered sets:

(a) Let X be a set. Then $M = \mathcal{P}(X)$, the power of X, with the relation $\leq = \subseteq$ is a poset.

(b) Let X be a set. Then $M = \{\mathcal{A} \mid \mathcal{A} \text{ is a topology on } X\}$, with the relation $\leq =$ "be coarser than" is a poset.

(c) Let X be a set. Then $M = \{\mathcal{F} \mid \mathcal{F} \text{ is a filter in } X\}$ with the relation $\leq = \subseteq$ is a poset.

(d) $M = \mathbb{R}$ and $\mathbb{Z}$ the usual order relation are posets.

7.3.3 Definition. Let M be a partially ordered set. An element $a \in M$ is called *maximal* if for no $b \neq a$, $a \leq b$.

7.3.4 Examples. In Examples 7.3.2 the following elements are maximal:

(a) $X \in M$.

(b) The discrete topology on X.

(c) The filters that do admit filters which are strictly finer.

(d) There are no maximal elements in $\mathbb{R}$ nor in $\mathbb{Z}$.

7.3.5 Definition. Let M be a partially ordered set and take $A \subset M$. An *upper bound* of A is an element $b \in M$ such that $a \leq b$ for all $a \in A$. If such an upper bound of A exists, then A is said to be *bounded from above*.

7.3.6 Examples. In Examples 7.3.2 we have the following:

(a) y (b) Every subset of M is bounded from above.

(c) Not every set of filters is bounded from above.

(d) Not every subset is bounded from above, for instance M itself.

However, we have the following result.

7.3.7 **Theorem**. *Let M be the set of all filters in a set X with the order relation $\subseteq$. Each totally ordered subset of M has an upper bound.*

Proof: Let $\mathcal{A}$ be a totally ordered set of filters in X and take $\mathcal{F}_0 = \bigcup_{\mathcal{F} \in \mathcal{A}} \mathcal{F}$. It is straightforward to verify that $\mathcal{F}_0$ is a filter, which is obviously an upper bound of $\mathcal{A}$. □

The following is an important result in Set Theory, whose proof we omit (see [9]).

7.3.8 **Theorem**. (Zorn's lemma) *Let M be a nonempty partially ordered set. If each totally ordered subset of M has an upper bound, then for every $a \in M$ there is a maximal element $b \in M$ such that $a \leq b$.* □

7.3.9 Definition. A maximal filter in a set X is called *ultrafilter*.

As a consequence of 7.3.7 and 7.3.8, we have the following result.

7.3.10 **Theorem**. *Let X be a set. Then for each filter $\mathcal{F}$ in X, there is an ultrafilter $\mathcal{U}$ which is finer than $\mathcal{F}$.* □

7.3.11 EXAMPLE. Let X be a set and take $x \in X$. Then the filter $\mathcal{F}_x = \{A \subset X \mid x \in A\}$ is an ultrafilter. Indeed, these are the only ultrafilters which can be explicitly described.

7.3.12 EXAMPLE. Let X a set and take an infinite sequence (x_n). Then the associated elementary filter $\mathcal{F}_{(x_n)}$ is not an ultrafilter, since all proper subsequences determine elementary filters, which are strictly finer than $\mathcal{F}_{(x_n)}$.

We saw in Theorem 7.2.17 that for every filter $\mathcal{F}$ which has a countable basis, there is an elementary filter $\mathcal{F}'$ finer than $\mathcal{F}$. We also observed in the previous example that the elementary filters are almost never ultrafilters. Consequently, the ultrafilters are either as $\mathcal{F}_x$ or do not admit countable bases. We have the following.

7.3.13 **Theorem**. *Let X be a set. Then a filter $\mathcal{U}$ in X is an ultrafilter if and only if for each $A \subset X$ one has either $A \in \mathcal{U}$ or $X - A \in \mathcal{U}$.*

Proof: Assume first that $\mathcal{U}$ is an ultrafilter. If $X - A \notin \mathcal{U}$, then by 7.1.25 we know that there is a filter $\mathcal{F}$ which is finer than $\mathcal{U}$ such that $A \in \mathcal{F}$. But since $\mathcal{U}$ is an ultrafilter $\mathcal{F} = \mathcal{U}$.

Conversely, if $\mathcal{U}$ is not an ultrafilter, then there is an ultrafilter $\mathcal{F}$, which is finer than and different from $\mathcal{U}$. Take $A \in \mathcal{F} - \mathcal{U}$. Clearly $A,\ X - A \notin \mathcal{U}$, since if $X - A \in \mathcal{U} \subset \mathcal{F}$, then we would have $X - A,\ A \in \mathcal{F}$, which is impossible. □

7.3.14 **Corollary**. *Let X be a set and let $\mathcal{U}$ be an ultrafilter in X. If $A_1, \dots, A_n \subset X$ are such that $\bigcup_{i=1}^n A_i \in \mathcal{U}$, then $A_i \in \mathcal{U}$ for some $i \in \{1, \dots, n\}$.*

Proof: If $A_i \notin \mathcal{U}$ for all i, then by 7.3.13, $X - A_i \in \mathcal{U}$ for all i and therefore

$$\bigcap_{i=1}^{n}(X - A_i) = X - \bigcup_{i=1}^{n} A_i \in \mathcal{U}\,,$$

which is a contradiction. □

If in the previous corollary we take $A_1 = A$ and $A_2 = X - A$, then we recover Theorem 7.3.13. Hence 7.3.13 and 7.3.14 are equivalent statements, and 7.3.14 is also a characterization of the ultrafilters. In other words, we have the following.

7.3.15 **Theorem**. *Let X be a set. A filter $\mathcal{U}$ in X is an ultrafilter if and only if each time that $A_1, \ldots, A_n \subset X$ are such that $\bigcup_{i=1}^{n} A_i \in \mathcal{U}$, then $A_i \in \mathcal{U}$ for some $i \in \{1, \ldots, n\}$.* □

7.3.16 EXERCISE. Show that every filter $\mathcal{F}$ in a set X is the intersection of all ultrafilters $\mathcal{U}$ that contain $\mathcal{F}$.

7.3.17 EXERCISE. Let $\mathcal{F}$ be a filter in a set X such that there is a unique ultrafilter $\mathcal{U}$ which is finer than $\mathcal{F}$. Show that $\mathcal{F} = \mathcal{U}$.

7.3.18 EXERCISE. Let X be a topological space and take $A \subset X$.

(a) Show that $x \in A^\circ$ if and only if $A \in \mathcal{F}$ for every filter $\mathcal{F}$ that converges to x.

(b) Show that $x \in \overline{A}$ if and only if there is at least one filter $\mathcal{F}$ that converges to x, such that $A \in \mathcal{F}$.

7.3.19 EXERCISE. Let X be a topological space. Show that an ultrafilter $\mathcal{U}$ in X either converges or does not have cluster points.

7.3.20 EXERCISE. Let $\mathcal{P}$ be a set of subsets of a set X such that if $P_1, P_2 \in \mathcal{P}$, then $P_1 \cap P_2, P_1 \cup P_2 \in \mathcal{P}$. A *$\mathcal{P}$-filter* is a nonempty family $\mathcal{F}$ of nonempty sets in $\mathcal{P}$ such that

(i) $P_1, P_2 \in \mathcal{F} \Longrightarrow P_1 \cap P_2 \in \mathcal{F}$, and

(ii) $P_1 \in \mathcal{F}, P_1 \subset P_2 \in \mathcal{P} \Longrightarrow P_2 \in \mathcal{F}$.

A *$\mathcal{P}$-ultrafilter* is a maximal $\mathcal{P}$-filter. If X is a topological space, then an *open filter* in X is a $\mathcal{P}$-filter such that $\mathcal{P}$ is the set of open sets in X (i.e. its topology) and an *open ultrafilter* is a maximal open filter.

(a) Show that every open filter is contained in an open ultrafilter.

(b) Show that the next statements are equivalent:

  (1) $\mathcal{U}$ is un open ultrafilter.

  (2) If G is an open set in X such that $G \cap H \neq \emptyset$ for all $H \in \mathcal{U}$, then $G \in \mathcal{U}$.

  (3) If G is an open set in X and $G \notin \mathcal{U}$, then $X - \overline{G} \in \mathcal{U}$.

7.3.21 EXERCISE.

(a) Let $\mathcal{F}$ be a filter in a set X such that $A = \bigcap_{F\in\mathcal{F}} F \neq \emptyset$. What relation do the filter $\mathcal{F}$ and the filter $\mathcal{F}_A$ generated by A keep?

(b) Does a filter $\mathcal{F}$ exist such that $\bigcap_{F\in\mathcal{F}} F = \emptyset$?

7.4 FILTERS AND FUNCTIONS

Let X and X' be sets and take a function $f : X \longrightarrow X'$. If $\mathcal{B}$ is a filter basis in X, then the images under f of the elements of $\mathcal{B}$ make up a filter basis in X', since if we take $B, B' \in \mathcal{B}$, then there is a set $B'' \in \mathcal{B}$ such that $B'' \subset B \cap B'$. Hence $f(B'') \subset f(B) \cap f(B')$.

If $\mathcal{F}$ is a filter in X, then in general the image under f of its elements does not form a filter. For instance, if f is not surjective, then X' is not one of the images. However, given that any filter is a filter basis, the images of the elements of $\mathcal{F}$ under f also make up a filter basis.

7.4.1 DEFINITION. Let $f : X \longrightarrow X'$ be a function of sets, and let $\mathcal{F}$ be a filter in X. The filter generated by the images under f of the elements of $\mathcal{F}$ is called the *image* of $\mathcal{F}$ under f and is denoted simply by $f(\mathcal{F})$.

7.4.2 EXAMPLE. Let $\mathbb{N}$ have the usual order and take $E_n = \{k \in \mathbb{N} \mid k \geq n\}$. The sets E_n constitute a filter basis in $\mathbb{N}$ and the filter $\mathcal{E}$ generated by this filter basis is the filter whose elements are the sets in $\mathbb{N}$ with finite complement. A function $\mathbb{N} \longrightarrow X$ is precisely a sequence (x_n) in X and the image of $\mathcal{E}$ under this function is the elementary filter $\mathcal{F}_{(x_n)}$ associated to this sequence.

Take a filter basis $\mathcal{B}'$ in the codomain X' of f. The inverse images under f of the elements of $\mathcal{B}'$ satisfy the axiom (FB), since $f^{-1}(A' \cap B') = f^{-1}(A') \cap f^{-1}(B')$. If we wish that these inverse images form a filter basis, we need that for every $B' \in \mathcal{B}'$, one has $f^{-1}(B') \neq \emptyset$. A sufficient condition for this is that f is surjective.

If $\mathcal{F}'$ is now a filter in X' and $f^{-1}(F') \neq \emptyset$ for all $F' \in \mathcal{F}'$, then $\{f^{-1}(F') \mid F' \in \mathcal{F}'\}$ is a filter basis, but not no necessarily a filter. Namely, if $F \supset f^{-1}(F')$, then there is not always a set $F'_1 \in \mathcal{F}'$ such that $F = f^{-1}(F'_1)$. (Why? *Exercise*).

7.4.3 DEFINITION. Let $f : X \longrightarrow X'$ be a set function and let $\mathcal{F}'$ be a filter in X'. If the inverse images under f of the elements of $\mathcal{F}'$ form a filter basis in X, then the filter generated by this filter basis is called *inverse image* of $\mathcal{F}'$ under f and is denoted by $f^{-1}(\mathcal{F}')$.

We have the following results.

7.4.4 **Theorem**. *Let $f : X \longrightarrow X'$ be a function of sets and let $\mathcal{B}'$ be a filter basis in X'. The filter $\mathcal{F}'$ generated by $\mathcal{B}'$ has an inverse image under f if and only if $f^{-1}(B') \neq \emptyset$ for each $B' \in \mathcal{B}'$.* □

7.4.5 **Theorem**. *Let $f : X \longrightarrow X'$ be a function of sets. If $\mathcal{F}' = f(\mathcal{F})$ for some filter $\mathcal{F}$ in X, then there is an $f^{-1}(\mathcal{F}')$. Moreover, if f is surjective, then $f^{-1}(\mathcal{F}')$ exists for any $\mathcal{F}'$.* □

Take $A \subset X$ and $A' \subset X'$. If $f : X \longrightarrow X'$ is a function, then one has that $A \subset f^{-1}(f(A))$ and that $f(f^{-1}(A')) \subset A'$. In particular, if f is injective, then $A = f^{-1}(f(A))$ and if f is surjective, then $f(f^{-1}(A')) = A'$. Hence we have the following.

7.4.6 **Theorem**. *Let $f : X \longrightarrow X'$ be a function of sets.*

(a) *If $\mathcal{F}$ is a filter in X, then $f^{-1}(f(\mathcal{F}))$ is a coarser filter than $\mathcal{F}$ and coincides with $\mathcal{F}$ if f is injective.*

(b) *If $\mathcal{F}'$ is a filter in X' and $f^{-1}(\mathcal{F}')$ exists, then $f(f^{-1}(\mathcal{F}'))$ is a finer filter than $\mathcal{F}'$ and coincides with $\mathcal{F}'$ if f is surjective.* □

7.4.7 **Proposition**. *Let $f : X \longrightarrow X'$ be a function of sets. If $\mathcal{G}$ is a finer filter than $\mathcal{F}$ in X, then $f(\mathcal{G})$ is finer than $f(\mathcal{F})$. Moreover, if $\mathcal{F}'$ is a finer filter than $\mathcal{G}'$ in X', then $f^{-1}(\mathcal{F}')$ is finer than $f^{-1}(\mathcal{G}')$, in case that these exist.* □

7.4.8 Note. If one replaces the word "finer by "strictly finer in the previous theorem, then it does not hold anymore. (*Exercise*).

7.4.9 Note. If $\mathcal{B}$ is a basis of $\mathcal{F}$, then $f(\mathcal{B}) = \{f(B) \mid B \in \mathcal{B}\}$ is a basis of $f(\mathcal{F})$. If on the other hand, $\mathcal{B}'$ is a basis of $\mathcal{F}'$, then $f^{-1}(\mathcal{B}') = \{f^{-1}(B') \mid B' \in \mathcal{B}'\}$ is a basis of $f^{-1}(\mathcal{F}')$. Of course both statements only have sense if the corresponding filters do exist.

7.4.10 **Proposition**. *Let $f : X \longrightarrow Y$ be a surjective function of sets and let $\mathcal{U}$ be an ultrafilter in X, then $f(\mathcal{U})$ is un ultrafilter en Y.*

Proof: Let $\mathcal{U}$ be an ultrafilter in X. Take $A \subset Y$. Then $f^{-1}(Y - A) = X - f^{-1}(A)$ and hence, by 7.3.13, $f^{-1}(A)$ or $f^{-1}(Y - A)$ is an element of $\mathcal{U}$, i.e. either A or $Y - A$ is an element of $f(\mathcal{U})$ and, again by 7.3.13, $f(\mathcal{U})$ is an ultrafilter. □

7.4.11 DEFINITION. Take $A \subset X$. Each filter $\mathcal{G}$ in A has an image in X under the inclusion $A \hookrightarrow X$, which is called *continuation* (or *extension*) of $\mathcal{G}$ to X.

If X is a topological space and $\mathcal{G}$ is a filter in the subspace A of X such that its continuation to X converges in X, we shall simply say that $\mathcal{G}$ converges in X. This does not necessarily mean that $\mathcal{G}$ converges in A. However, if $\mathcal{G}$ converges in A, then also its continuation to X converges (to the same point of A). More concretely, one can solve the following exercise.

7.4.12 EXERCISE. Let X be topological space and take $A \subset X$ and a filter $\mathcal{G}$ in A. Show

(a) A point $x \in A$ is a cluster point of $\mathcal{G}$ if and only if x is a cluster point of the continuation of $\mathcal{G}$ to X.

(b) $x \in A$ is a limit point of $\mathcal{G}$ if and only if x is a limit point of the continuation of $\mathcal{G}$ to X.

If $\mathcal{F}$ is a filter in a set X, as we already saw, the filter $i^{-1}(\mathcal{F})$ need not exist if i is the inclusion of A in X. Indeed, $i^{-1}(\mathcal{F})$ exists if and only if $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$. In this case the filter $i^{-1}(\mathcal{F})$ is called the *trace* of $\mathcal{F}$ in A. If $A \notin \mathcal{F}$, but the trace of $\mathcal{F}$ in A exists, then the continuation of the trace is the supremum of $\mathcal{F}$ and the filter $\mathcal{F}_A$ of all supersets of A. If, in particular, $\mathcal{F}$ is the neighborhood filter of a point $x \in X$, then from the definition of limit point of a set in a topological space, we obtain the following result, which generalizes Theorem 7.1.4 (a).

7.4.13 **Theorem**. *Take $A \subset X$ and $x \in X$. The following are equivalent:*

(a) $x \in \overline{A}$.

(b) *The neighborhood filter of x has a trace in A.*

(c) *There is a filter $\mathcal{G}$ in A such that $\mathcal{G} \to x$ in X.* □

Let X and Y be topological spaces and take a function $f : X \longrightarrow Y$ and a point $x_0 \in X$. In terms of filters, what does it mean that f is continuous in x_0? By definition, this means that given a neighborhood of $f(x_0)$, there is a neighborhood of x_0 whose image lies in the given neighborhood, namely $\mathcal{N}_{f(x_0)} \subset f(\mathcal{N}_{x_0})$. In other words, we have the following result.

7.4.14 **Proposition**. *A map $f : X \longrightarrow Y$ is continuous at x_0 if and only if $f(\mathcal{N}_{x_0}) \to f(x_0)$.* □

In case that this limit is unique, then we simply write

$$\lim_{x \to x_0} f(x) = f(x_0)\,. \tag{7.4.15}$$

Hence, if Y is a Hausdorff space, then this equation characterizes the continuity of f at x_0.

Now, if we have that f is continuous at x_0 and $\mathcal{F}$ is a filter that converges to x_0, then $\mathcal{F}$ is finer than $\mathcal{N}_{x_0}$, so that $f(\mathcal{F})$ is finer than $f(\mathcal{N}_{x_0})$. This filter is finer than $\mathcal{N}_{f(x_0)}$. Hence we have the following result, which generalizes Theorem 7.1.4 (b).

7.4.16 **Proposition**. *A map $f : X \longrightarrow Y$ is continuous at x_0 if and only if $f(\mathcal{F}) \to f(x_0)$ for every filter $\mathcal{F}$ such that $\mathcal{F} \to x_0$.* □

In other words, continuous maps are precisely those maps which commute with limits, namely, such that $f(\lim \mathcal{F}) = \lim f(\mathcal{F})$. In particular we obtain Theorem 7.1.4 (b) as a consequence.

We finish this section analyzing another fundamental concept in topology, which we already considered above 3.4.12 namely the concept of density. We start with a more general definition.

7.4.17 DEFINITION. Let X be a topological space and take $A, B \subset X$. We say that A is *dense with respect to* B if $B \subset \overline{A}$. We say that A is *dense in* B if $A \subset B \subset \overline{A}$.

7.4.18 **Lemma**. *If A is dense in X and Q is open in X, then $A \cap Q$ is dense in Q.*

Proof: We wish to show that $Q \subset \overline{A \cap Q}$. For this take $x \in Q$ and a neighborhood U of x Since Q is open, $U \cap Q$ is a neighborhood of x, and since A is dense in X, then $U \cap Q \cap A \neq \emptyset$. Therefore $x \in \overline{Q \cap A}$. □

We have that if X is a Hausdorff space, then by 7.4.14, $f : X \longrightarrow Y$ is continuous at $x_0 \in X$ if and only if $\lim_{x \to x_0} f(x) = f(x_0)$, namely, by definition, if and only if $f(\mathcal{N}_{x_0}) \to f(x_0)$.

If $\mathcal{N}_{x_0}$ induces a filter in A whose image under $f|_A$ converges to y, then we may write that

$$\lim_{\substack{x \to x_0 \\ x \in A}} f(x) = y\,.$$

In particular, if $A = X - \{x_0\}$, then we write

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = y\,.$$

Assume that A is dense in X and that Y is a Hausdorff space. Then for any $x_0 \in X$, the trace of $\mathcal{N}_{x_0}$ in A exists. If $f : X \longrightarrow Y$ is continuous, then take $g = f|_A$, so that

$$\lim_{\substack{x \to x_0 \\ x \in A}} g(x) = f(x_0)\,.$$

We have the next result.

7.4.19 **Theorem**. *Let X and Y be topological spaces, Y Hausdorff, and take $A \subset X$ dense. Let $f, g : X \longrightarrow Y$ be continuous maps. If $f|_A = g|_A$, then $f = g$.* □

7.4.20 EXERCISE. Prove the previous theorem directly using explicitly the facts that $\overline{A} = X$ y and that Y is Hausdorff.

7.4.21 EXERCISE. Let $f : X \longrightarrow Y$ be a continuous map and let $D \subset X$ be a dense subset. Show that $f(D)$ is dense in $f(X) \subset Y$. In particular, if f is surjective, then $f(D)$ is dense in Y. (*Hint:* Use 2.5.8 (d).)

7.5 FILTERS AND PRODUCTS

Filters are a good tool for studying the product topology. En this section we shall analyze the behavior of filters with respect to products. We start with the next result, which characterizes the filter convergence in a product.

7.5.1 **Theorem**. *Take a product of topological spaces $X = \prod_{\lambda \in \Lambda} X_\lambda$ and let $\mathcal{F}$ be a filter in X. Then $\mathcal{F} \to x = (x_\lambda)$ if and only if, for all $\lambda \in \Lambda$, each image filter $p_\lambda(\mathcal{F}) \to x_\lambda$, where $p_\lambda : X \longrightarrow X_\lambda$ is the projection.*

Proof: Since p_λ is continuous, if $\mathcal{F}$ converges to x, then $p_\lambda(\mathcal{F})$ converges to $p_\lambda(x)$.

Conversely, assume that $p_\lambda(\mathcal{F}) \to x_\lambda$ for all λ, and let Q_λ be an open neighborhood of x_λ in X_λ. Therefore $Q_\lambda \in p_\lambda(\mathcal{F})$ and thus $p_\lambda^{-1}(Q_\lambda) \in p_\lambda^{-1} p_\lambda(\mathcal{F}) \subset \mathcal{F}$. Consequently $\mathcal{F}$ contiains also finite intersections of elements of the form $p_\lambda^{-1}(Q_\lambda)$, namely sets of the form $\prod_{\lambda \in \Lambda} Q_\lambda$, where $Q_\lambda = X_\lambda$, except for finitely many $\lambda \in \Lambda$. Since these sets form a neighborhood basis of $x = (x_\lambda)$ in X, we have that $\mathcal{N}_x \subset \mathcal{F}$, i.e. $\mathcal{F} \to x$. □

7.5.2 EXAMPLE. Let $\mathbb{R}^{\mathbb{R}}$ be the product of copies of $\mathbb{R}$, one for each $t \in \mathbb{R}$. In other words, $\mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \longrightarrow \mathbb{R}\}$ with the product topology. Then $f_n \to f$ in $\mathbb{R}^{\mathbb{R}}$ if and only if $f_n(t) \to f(t)$ for all $t \in \mathbb{R}$. In other words, one has function convergence in the product topology if and only if one has pointwise convergence.

Theorem 7.5.1 provides a characterization of the product topology by stating which filters are convergent. We ask now generally about the existence of filters in the cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$ of sets X_λ with given projections. We already suggested in Example 7.1.8(d) how to construct a filter basis in a product, given a filter basis in each factor.

Thus let $\mathcal{F}_\lambda$ be a filter in a set X_λ, $\lambda \in \Lambda$. Take $\kappa \in \Lambda$ and $F_\kappa \in \mathcal{F}_\kappa$. Then $p_\kappa^{-1}(F_\kappa) = \prod_{\lambda\in\Lambda} F_\lambda$, where $F_\lambda = F_\kappa$ if $\lambda = \kappa$ and $F_\lambda = X_\lambda$ if $\lambda \neq \kappa$. The supremum of $\{p_\lambda^{-1}(\mathcal{F}_\lambda) \mid \lambda \in \Lambda\}$ exists and is generated, precisely, by $\{\prod_{\lambda\in\Lambda} F_\lambda \mid F_\lambda \in \mathcal{F}_\lambda$, and $F_\lambda = X_\lambda$, except for finitely many $\lambda \in \Lambda\}$.

7.5.3 DEFINITION. Let $\Lambda \neq \emptyset$ and X_λ be a nonempty set for each $\lambda \in \Lambda$. The filter $\mathcal{F}$ generated by the filter basis

$$\{\prod_{\lambda\in\Lambda} F_\lambda \mid F_\lambda \in \mathcal{F}_\lambda, \text{ and } F_\lambda = X_\lambda, \text{ except for finitely many } \lambda \in \Lambda\}$$

is called *product* of the filters $\mathcal{F}_\lambda$ and is denoted by $\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda$ and is such that $p_\lambda(\mathcal{F}) = \mathcal{F}_\lambda$. Compare with 7.1.8(d).

Take now nonempty topological spaces X_λ, $\lambda \in \Lambda$, and let $\mathcal{F}_\lambda$ be a filter in X_λ for each λ. If $\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda$ has two different limits in $\prod_{\lambda\in\Lambda} X_\lambda$, then $\mathcal{F}_\lambda$ also has two different limits in X_λ para some λ. Conversely, if for some λ there is a filter $\mathcal{F}_\lambda$ which has two different limits in X_λ, then (if for each λ, $X_\lambda \neq \emptyset$) there is a filter $\mathcal{F}$ with two different limits in $X = \prod_{\lambda\in\Lambda} X_\lambda$. Thus we have, as a first application of filters to products of topological spaces, the following.

7.5.4 **Theorem**. *Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a nonempty family of nonempty topological spaces and take $X = \prod_{\lambda\in\Lambda} X_\lambda$. Then X is Hausdorff if and only if X_λ is Hausdorff for every $\lambda \in \Lambda$.* □

Take $\Lambda \neq \emptyset$ and for each $\lambda \in \Lambda$, take a nonempty set X_λ. Let $\mathcal{U}$ be an ultrafilter in cartesian the product $X = \prod_{\lambda\in\Lambda} X_\lambda$. Take $\mathcal{U}_\lambda = p_\lambda(\mathcal{U})$ and let $\mathcal{F}_\lambda$ be a filter in X_λ, which is finer than $\mathcal{U}_\lambda$. Then $\mathcal{F} = \prod_{\lambda\in\Lambda} \mathcal{F}_\lambda$ is a finer filter than $\mathcal{U}$. But since $\mathcal{U}$ is an ultrafilter, $\mathcal{F} = \mathcal{U}$. Due to the fact that $p_\lambda(\mathcal{F}) = \mathcal{F}_\lambda$ and $p_\lambda(\mathcal{U}) = \mathcal{U}_\lambda$, we conclude that $\mathcal{F}_\lambda = \mathcal{U}_\lambda$. Therefore $\mathcal{U}_\lambda$ is also an ultrafilter.

Conversely, if $\mathcal{U}_\lambda$ is an ultrafilter for each $\lambda \in \Lambda$, then take $\mathcal{U} = \prod_{\lambda\in\Lambda} \mathcal{U}_\lambda$ and let $\mathcal{F}$ be a finer filter in X than $\mathcal{U}$. Hence $p_\lambda(\mathcal{F})$ is finer than $p_\lambda(\mathcal{U}) = \mathcal{U}_\lambda$. Hence $p_\lambda(\mathcal{F}) = \mathcal{U}_\lambda$ and consequently $\mathcal{F} = \mathcal{U}$. Thus we have tha next result.

7.5.5 **Theorem**. *Take a nonempty family* $\{X_\lambda \mid \lambda \in \Lambda\}$ *of nonempty sets. Then* $\mathcal{U} = \prod_{\lambda\in\Lambda} \mathcal{U}_\lambda$ *is an ultrafilter if and only if* $\mathcal{U}_\lambda$ *is un ultrafilter for each* $\lambda \in \Lambda$. □

7.5.6 Exercise. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a nonempty family of nonempty topological spaces and for each $\lambda \in \Lambda$, let $\mathcal{F}_\lambda$ be a filter in X_λ. Let $\mathcal{F}$ the product filter in $X = \prod_{\lambda\in\Lambda}$. Prove

(a) $x = (x_\lambda) \in X$ is a cluster point of $\mathcal{F}$ if and only if $x_\lambda \in X_\lambda$ is a cluster point of $\mathcal{F}_\lambda$ for all λ.

(b) $\mathcal{F} \to x = (x_\lambda) \in X$ if and only if $\mathcal{F}_\lambda \to x_\lambda \in X_\lambda$ for all λ.

If $p_\lambda = \mathrm{proj}_{X_\lambda} : X \longrightarrow X_\lambda$, then conclude that for an arbitrary filter $\mathcal{F}$ in X the following hold:

(c) If $x_\lambda \in X_\lambda$ is a cluster point of $\mathcal{F}_\lambda = p_\lambda(\mathcal{F})$, then $x = (x_\lambda) \in X$ is a cluster point of $\mathcal{F}$,

(d) If $\mathcal{F}_\lambda = p_\lambda(\mathcal{F}) \to x_\lambda \in X_\lambda$, then $\mathcal{F} \to x = (x_\lambda) \in X$.

7.6 Nets

When a topological space is first-countable, we have shown that sequence-convergence characterizes its topology. Nets are a generalization of sequences and their convergence can be used to characterize the topology of arbitrary spaces in a very similar way as sequence-convergence does in the special case of first-countable spaces. In contrast to filters, nets behave in a very similar way to sequences and therefore are easy to use. In some sense, nets and filters are equivalent: Given a net, there is a filter whose convergence corresponds to that of the net. Conversely, given a filter, one can associate a net, with similar convergence. In this section we shall prove the most important results about nets and topological spaces using the relation ship between nets and filters.

7.6.1 Definition. A set M is called a *directed set* if it has a *preorder relation* $\leq$ that satisfies the first two order axioms (OR1) (the relation is reflexive) and (OR2) (the relation is transitive) of 7.3.1, and the following additional axiom:

(DS) For any $a, b \in M$ there exists $c \in M$ such that $a \leq c$ and $b \leq c$.

7.6.2 Examples. The following are directed sets:

(a) $M = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{N}$ with the usual order relation $\leq$.

(b) $M = \mathcal{P}(X)$ the power set of a set X with the preorder relation $\leq\, = \,\supseteq$ (or $\leq\, = \,\subseteq$).

(c) $M = \mathcal{N}_x$, a local neighborhood basis at a point x in a topological space X, with the preorder relation $\leq\, = \,\supset$ (or $\leq\, = \,\subset$).

The relation $\leq$ is called a *direction* in M.

In order to study convergence in an adequate set up in a topological space, the directed set of Example (c) will be needed. The next definition generalizes sequences.

7.6.3 Definition. Let X be a set and M be a directed set. An M*-net*, or simply a *net* in X is a function $M \longrightarrow X$. For each $a \in M$ we denote by $x_a \in X$ the image of a in X and we represent the net by $(x_a)_{a\in M}$, or simply by (x_a).

Let M' be a directed set and $\varphi : M' \longrightarrow M$ an *increasing* and *cofinal* function, that is, it satisfies

$$a' \leq b' \implies \varphi(a') \leq \varphi(b') \quad (\varphi \text{ is } \textit{increasing})$$

and

$$\text{given } a \in M, \text{ there is some } a' \in M' \text{ such that } a \leq \varphi(a') \quad (\varphi \text{ is } \textit{cofinal}).$$

We define a *subnet* of $(x_a)_{a\in M}$ as the M'-net $(x_{\varphi(a')})_{a'\in M'}$.

In analogy to sequences, one can define the *tail* of (x_a) as the set $(x_a)_b = (x_a)_{b\leq a}$, for some $b \in M$. Since given $b_1, b_2 \in M$, there exists $b \in M$ such that $b_1, b_2 \leq b$, one has that $\{x_a\}_b \subset \{x_a\}_{b_1} \cap \{x_a\}_{b_2}$. Hence, the set of tails of a net is a filter basis which generates a filter $\mathcal{F}_{(x_a)}$, which is called *filter generated by the net* (x_a).

Since net-convergence is modeled after sequence-convergence, the following definitions should be clear.

7.6.4 Definition. Let $(x_a)_{a\in M}$ be a net in X and let $A \subset X$. We say that the net (x_a) lies *frequently* in A if for every $a \in M$ there exists $b \in M$, $a \leq b$ such that $x_b \in A$ and that it lies *finally* in A if there exists $a \in M$ such that for every $b \in M$, $a \leq b$, $x_b \in A$.

Assume now that X is a topological space. A net (x_a) *converges* to a point $x \in X$ (in symbols, $(x_a) \to x$ or simply $x_a \to x$) if for any neighborhood V of x

the net lies finally in V. Then one defines x as a *limit point* of the net (in symbols, $x \in \lim x_a$). One says that y is an *cluster point* (or *accumulation point*) of a net (x_a) if for any neighborhood W of y the net lies frequently in W.

If, for instance, X is a discrete space, then a net (x_a) converges to $x \in X$ if and only if (x_a) lies *finally* in $\{x\}$, namely if and only if there is b such that if $b \leq a$, then $x_a = x$. On the other hand, if X is indiscrete (with more than one point), then every net (x_a) converges to any point in X, that is, $\lim x_a = X$. Thus, a net may have several limit points.

7.6.5 EXERCISE. Prove that if a net (x_a) in X lies finally in a set $V \subset A$, then it also lies frequently in it. Conclude that any limit point of a net in a topological space is also an cluster point of the net.

7.6.6 EXERCISE. Prove that if a net (x_a) in a topological space X is finally constant (say finally x), then (x_a) converges (namely $x_a \to x$).

7.6.7 EXAMPLES.

(a) Let X be a topological space, $x \in X$, and let M be any local neighborhood basis in X at the point x. As in 7.6.2(c), we give M a relation by $U \leq V$ if and only if $V \subset U$, that makes it to a directed set. Take a point $x_U \in U$ for every $U \in M$. Then (x_U) is a net in X such that $x_U \to x$. Namely, given any neighborhhod W of x in X, there is some $U \in M$ such that $U \subset W$. Therefore, if $U \leq V$ in M, then $V \subset W$, and so $x_V \in W$. Thus (x_U) lies finally in the given neighborhood W.

(b) Since the set $\mathbb{N}$ of the natural numbers with their standard order relation is a directed set, every sequence (x_n) is a net. Clearly, (x_n) converges to x as a sequence if and only if it does converge to x as a net. Observe that every subsequence of (x_n) is a subnet of the net (x_n); however, the converse is not true. One may give a subnet of (x_n) that is not a subsequence.

(c) Recall that a *partition* of a closed interval $[a, b]$ is a finite sequence $P = \{t_0 = a < t_1 < t_2 < \cdots < t_k = b\}$ and that another partition Q is a *refinement* if $P \subset Q$. We write this fact by $P \leq Q$. Thus the set $\mathcal{P}$ of all partitions of $[a, b]$ is a directed set. Given any real-valued function f on $[a, b]$, we can define a net $L(f) : \mathcal{P} \longrightarrow \mathbb{R}$ by letting $L(f)(P)$ be the lower Riemann sum of f over the partition P and $U(f) : \mathcal{P} \longrightarrow \mathbb{R}$ by letting $U(f)(P)$ be the upper Riemann sum of f over the partition P (see [14, 6.1]). That both of these two nets converge to the same value c means that f is integrable, and

$$\int_a^b f(t)dt = c\,.$$

This example is what led Moore and Smith to the concept of a net (see [13]).

7.6.8 EXERCISE. Let X be a metric space and take $x_0 \in X$. Prove that the complement of this point $M = X - x_0$ becomes a directed set when preordered by the relation $x < x'$ if and only if $d(x', x_0) < d(x, x_0)$. Take $f : X \longrightarrow Y$, where Y is another metric space. The restriction $f|_M : M \longrightarrow Y$ defines a net in Y. Prove that this net converges to $y_0 \in Y$ if and only if $\lim_{x \to x_0} f(x) = z_0$ in the sense of elementary calculus.

7.6.9 EXERCISE. Prove that if a net (x_a) in a topological space X converges to x, then every subnet of (x_a) converges to x.

The following result relates net convergence and filter convergence.

7.6.10 **Proposition**. *Let X be a topological space and let (x_a) be a net in X. Then $x_a \to x$ if and only if $\mathcal{F}_{(x_a)} \to x$.*

Proof: If $x_a \to x$, then for each neighborhood $V \in \mathcal{N}_x$ there is a tail of the net (x_a) which is contained in V. Therefore $V \in \mathcal{F}_{(x_a)}$, i.e. $\mathcal{N}_x \subset \mathcal{F}_{(x_a)}$. Thus $\mathcal{F}_{(x_a)} \to x$.

Conversely, if $\mathcal{F}_{(x_a)} \to x$, namely, if $\mathcal{N}_x \subset \mathcal{F}_{(x_a)}$ and $V \in \mathcal{N}_x$, then V contains a tail of the net, that is, the net lies finally in V. Hence $x_a \to x$. □

Thus, since convergence for nets and for sequences are consistent concepts, so are the concepts of cluster point too. Namely, we have the next result, which generalizes Theorem 7.2.7 and whose proof is similar to the previous proof.

7.6.11 **Proposition**. *Let X be a topological space and take a net (x_a) in X. Then y is a cluster point of the net (x_a) if and only of if y is a cluster point of the filter $\mathcal{F}_{(x_a)}$ generated by the net.* □

From 7.2.13, the next assertion is immediate.

7.6.12 **Proposition**. *Let X be a Hausdorff space. Then every net (x_a) in X converges to at most one point $x \in X$.*

We can use nets to characterize the closure of a set in a topological space and with it, to characterize the topology.

7.6.13 **Theorem**. *Let X be a topological space and take $A \subset X$. The following are equivalent:*

(a) $x \in \overline{A}$.

(b) *There is a net* (x_a) *in* A *which converges to* x.

(c) *There is a net* (x_a) *in* A *which has* x *as cluster point.*

Proof:

(a) $\Longrightarrow$ (b) For each $V \in \mathcal{N}_x$, $V \cap A \neq \emptyset$ (since $x \in \overline{A}$). Let $M = \mathcal{N}_x$ be the directed set of Example 7.6.2(c) and let $M \to X$ be a net such that $x_V \in V \cap A$ for each $V \in M$. Thus the net (x_V) lies in A and for each $V \in \mathcal{N}_x$ one has that $x_W \in V$ if $V \leq W$, that is, if $V \supset W$. Therefore $x_V \to x$.

(b) $\Longrightarrow$ (c) Every limit point of a net is a cluster point, since if a net lies finally in some neighborhood, then it also lies frequently in it (see 7.6.5).

(c) $\Longrightarrow$ (a) Take $V \in \mathcal{N}_x$. Since x is a cluster point of a net (x_a) in A, the net lies frequently in V, i.e. for each a there is a b with $a \leq b$ such that $x_b \in V$. Therefore, since $x_b \in A$, we have $A \cap V \neq \emptyset$ so that $x \in \overline{A}$. □

As a consequence, we have the following characterization of the topology of a space using net convergence, which generalizes 7.1.4(a) to spaces that not necessarily are first-countable.

7.6.14 **Proposition**. *Let* X *be a topological space and take* $A \subset X$. *The following are equivalent:*

(a) A *is closed.*

(b) *For every net* (x_a) *in* A *such that* $y \in X$ *is a cluster point of it, one has* $y \in A$.

(c) *For every net* (x_a) *in* A *such that* $x \in X$ *is a limit point of it, namely* $x_a \to x$, *one has* $x \in A$.

Proof:

(a) $\Longrightarrow$ (b) By Proposition 7.6.13, $y \in \overline{A}$, so that, by (a), $y \in A$.

(b) $\Longrightarrow$ (c) Every limit point of a net is a cluster point. Thus $x \in A$.

(c) $\Longrightarrow$ (a) Take $x \in \overline{A}$. By 7.6.13, there is a net (x_a) in A which converges to x. Hence, by (c), $x \in A$, so that $\overline{A} \subset A$, namely, A is closed. □

Let $f : X \longrightarrow Y$ be a map between topological spaces and take a net $(x_a)_{a \in M}$ in X. Then there is a net $(f(x_a))_{a \in M}$ in Y, which we call *image* of the net $(x_a)_{a \in M}$ under f. Convergence of nets allows to characterize continuity. We have the next result.

7.6.15 **Proposition**. *A map between topological spaces $f : X \longrightarrow Y$ is continuous in a point $x_0 \in X$ if and only if, for every net which converges to x_0, $x_a \to x_0$, the image net converges to $f(x_0)$, $f(x_a) \to f(x_0)$.*

Proof: Assume first that $x_a \to x_0$ and take $V \in \mathcal{N}^Y_{f(x_0)}$. Since f is continuous at x_0, there is a neighborhood $U \in \mathcal{N}^X_{x_0}$ such that $f(U) \subset V$. Since the net converges, it lies finally in U. Thus its image $(f(x_a))$ lies finally in $f(U)$ and hence in V. Therefore $f(x_a) \to f(x_0)$.

Conversely, if f is not continuous at x_0, there is a neighborhood $V \in \mathcal{N}^Y_{f(x_0)}$ such that for all neighborhoods $U \in \mathcal{N}^X_{x_0}$, $f(U) \not\subset V$, namely, $f(U) \cap (Y - V) \neq \emptyset$. Since $M = \mathcal{N}^X_{x_0}$ is a directed set, then as we said in Example 7.6.2(c), we may define an M-net in X, (x_U) such that for each $U \in M$, the element $x_U \in U$ is such that $f(x_U) \notin V$. By construction, $x_U \to x_0$, but $f(x_U) \not\to f(x_0)$, since this image net does not lie finally in V. □

We put in one theorem, which generalizes Theorem 7.1.4, the previous results.

7.6.16 **Theorem**. *Let X be a topological space. Then*

(a) *$A \subset X$ is closed if and only if for every net (x_a) in A such that, if $x_a \to x_0$ in X, then $x_0 \in A$.*

(b) *$f : X \longrightarrow Y$ is continuous at $x_0 \in X$ if and only if for every net (x_a) in X such that $x_a \to x_0$, the image net $f(x_a) \to f(x_0)$.*

We thus have that the common techniques of sequences used for checking topological properties of first-countable spaces can be used for arbitrary spaces, provided that one takes nets instead of sequences.

7.6.17 EXERCISE. Let $\mathcal{F}$ be a filter in a set X.

(a) Put $M_{\mathcal{F}} = \{(x, F) \mid F \in \mathcal{F} \text{ and } x \in F\}$. Prove that with the order relation $(x, F) \leq (y, G) \iff F \supset G$, $M_{\mathcal{F}}$ is a directed set.

The $M_{\mathcal{F}}$-net in X, $M_{\mathcal{F}} \longrightarrow X$ such that $(x, F) \mapsto x$ is called the *net determined by the filter $\mathcal{F}$*.

(b) Prove that the filter determined by the $M_{\mathcal{F}}$-net such that $x_{(x,F)} = x$ is precisely $\mathcal{F}$.

(c) Let X be a topological space and take a filter $\mathcal{F}$ in X. Show that $\mathcal{F} \to x_0$ if and only if the net determined by $\mathcal{F}$ converges to x_0.

(d) Let M be an arbitrary directed set and take an M-net (x_a) in X. There is a function $M \longrightarrow M_{\mathcal{F}(x_a)}$ given by $a \mapsto (x_a, (x_b)_a)$. Show that this function preserves the order. Moreover, the net $(x, F) \mapsto x$ *extends* the given net (x_a), i.e. the composition

$$\begin{aligned} M \longrightarrow M_{\mathcal{F}} &\longrightarrow X \\ (x, F) &\mapsto x \end{aligned}$$

is the given net. Show that this last net converges to x_0 if and only if the given net converges to x_0

(e) Let $\mathcal{F}$ be a filter in X and let $M_{\mathcal{F}} \longrightarrow X$, $(x, F) \mapsto x$, be the net determined by $\mathcal{F}$. Prove that the filter $\mathcal{F}$ converges to x_0 if and only if the filter $\mathcal{F}(x_{(x,F)})$ converges to x_0.

The previous exercise shows that from the point of view of convergence, net theory and filter theory are equivalent.

7.6.18 Exercise. Let X be a topological space. Show that if a net (x_a) has a cluster point x, then the net has a subnet (y_b) which converges to x.

Chapter 8 Compactness

Compactness is one of the most important conditions needed to prove fundamental results in topology and analysis. En this chapter we shall study basic concepts related to compactness. We start analyzing several conditions equivalent to compactness and giving simple and deep applications of the concept. We shall study two fundamental theorems on compactness, namely the Heine–Borel–Lebesgue Theorem and the Bolzano–Weierstrass Theorem, as well as their generalizations to metric spaces and, more generally, to first-countable spaces.

Further on we shall study the concept of compactification, that refers to how one can densely embed noncompact spaces in compact spaces. This is a very important issue in several areas such as algebraic geometry and topology. We shall analyze here the one-point (or Alexandroff) compactification. In the next chapter we shall study the Stone–Čech compactification.

We shall finish the chapter we some applications of compactness to construct other classes of topological spaces which have nice properties and are useful for homotopy theory. These are the categories of compactly generated spaces and of k-spaces. We shall show to alter the topology of a given space to convert it into one of each of these classes. These changes of topology produce finer topologies in each case, however, the usual algebraic invariants studied in algebraic topology remain unchanged.

8.1 Compact sets

We start by briefly recalling the two main results on compactness in $\mathbb{R}^n$. For the time being, we shall omit the proof, but by the end of the chapter we shall give it.

8.1.1 **Theorem**. (Heine–Borel–Lebesgue) *In $\mathbb{R}^n$, a set A is closed and bounded if and only if every open cover of A contains a finite subcover.*

8.1.2 **Theorem**. (Bolzano–Weierstrass) *In $\mathbb{R}^n$, a set A is closed and bounded if and only if every sequence in A has an cluster point in A.*

Further on (8.2.12 and 8.2.13) we shall give the proof of these results, after we study the general theory of compactness.

8.1.3 Definition. Let X be topological space and $A \subset X$. We say that a family $\mathcal{C} = \{U_\lambda \subset X \mid \lambda \in \Lambda\}$ is a *cover* of A in X if the union $\bigcup_{\lambda\in\Lambda} U_\lambda \supset A$. We say that the cover $\mathcal{C}$ is *open*, resp. *closed*, if U_λ is open, resp. closed, in X for all $\lambda \in \Lambda$. If $\mathcal{C}' \subset \mathcal{C}$ and $\mathcal{C}'$ is again a cover of A, then we say that $\mathcal{C}'$ is a *subcover* of $\mathcal{C}$. In particular, we say that $\mathcal{C}'$ is finite, resp. countable, if as a set $\mathcal{C}'$ is finite, resp. countable.

8.1.4 **Theorem**. *Let X be a topological space. The following statements are equivalent:*

(C) *Every open cover of X contains a finite subcover.*

(C$'$) *Every family of closed sets in X, whose intersection is empty, contains a finite subfamily whose intersection is empty.*

(C$''$) *Given a family of closed sets in X such that any finite subfamily has an nonempty intersection, then the given family has nonempty intersection.*

(C$'''$) *Each filter in X has, at least, one cluster point.*

(C^{iv}) *Every ultrafilter in X converges.*

Proof:

(C)$\Longleftrightarrow$(C$'$) is clear since $\{U_\lambda \mid \lambda \in \Lambda\}$ is an open cover if and only if $\{X - U_\lambda \mid \lambda \in \Lambda\}$ is a family of closed sets with empty intersection.

(C$'$)$\Longleftrightarrow$(C$''$) is obvious.

(C$''$)$\Longrightarrow$(C$'''$) Let $\mathcal{F}$ be a filter in X. Each finite collection of closed sets $F \in \mathcal{F}$ has nonempty intersection, therefore, by (C$''$), $I = \bigcap\{F \mid F \in \mathcal{F},\ F \text{ is closed }\} \neq \emptyset$. Take $x \in I$, hence x is a cluster point of $\mathcal{F}$, since if $V \in \mathcal{N}_x$ and $G \in \mathcal{F}$, then $x \in \overline{G} \in \mathcal{F}$ and thus $V \cap G \neq \emptyset$.

(C$'''$)$\Longrightarrow$(C$''$) Let $\mathcal{G}$ be a family of closed sets such that $\bigcap_{i=1}^k G_i \neq \emptyset$, $G_i \in \mathcal{G}$, $k \in \mathbb{N}$, and let $\mathcal{F} = \{F \subset X \mid F \supset \bigcap_{i=1}^k G_i, G_i \in \mathcal{G}, k \in \mathbb{N}\}$. $\mathcal{F}$ is clearly a filter and by (C$'''$), it has a cluster point x. Therefore every neighborhood of x meets $\bigcap_{i=1}^k G_i$, $G_i \in \mathcal{G}$, $k \in \mathbb{N}$. Hence $x \in \overline{G}$ for each $G \in \mathcal{G}$, and thus $x \in \bigcap\{G \mid G \in \mathcal{G}\}$.

(C$'''$)$\Longleftrightarrow$(C^{iv}) is clear. □

8.1.5 Definition. A topological space X that satisfies one, and hence all, conditions (C),...,(C^{iv}), is said to be *compact*. A subset A of a topological space X is *compact* if as a subspace of X with the relative topology is a compact space.

8.1.6 **Proposition**. *Let X be a compact space. If a filter $\mathcal{F}$ in X has a unique cluster point $x \in X$, then $\mathcal{F}$ converges to x.*

Proof: If $\mathcal{F}$ did not converge to x, then there would be an open neighborhood V of x such that $V \notin \mathcal{F}$. Hence, by 7.1.25, there would be a filter $\mathcal{G}$ finer than $\mathcal{F}$ such that $X - V \in \mathcal{G}$. Since X is compact, $\mathcal{G}$ would have a cluster point $y \in X$. Since y belongs to $X - V$, y and x would be different. But y would be also a cluster point of $\mathcal{F}$, contardicting the uniqueness of the cluster point. □

8.1.7 EXAMPLES.

(a) Every indiscrete space is compact.

(b) A discrete space is compact if and only if it is finite.

(c) Every finite space is compact.

(d) The Cantor set defined in 2.2.7 is compact.

(e) Every space X with the cofinite topology is compact. Namely, if $\{U_\lambda\}$ is an open cover, then for any λ_1, the set $X - U_{\lambda_1}$ is finite. Hence we can choose a finite number of open sets $U_{\lambda_2}, \ldots, U_{\lambda_k}$ of the cover, whose union contains this finite set. Then $\{U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_k}\}$is a finite subcover of X

8.1.8 EXERCISE. Prove that the Cantor set C defined in 2.2.7 is homeomorphic to a product of a countable family of discrete spaces, each having two elements. This shows that the product of discrete spaces need not be discrete. (*Hint:* Each point $x \in I$ can be ternarily expressed as a sum $\sum \frac{x_i}{3^i}$, where $x_i \in \{0, 1, 2\}$. Thus, to each x corresponds an expression $(x_1, x_2, x_3, \ldots)$. These expressions are unique up to the fact that each number, except 1, whose ternary expression ends with a sequence of 2s, can be expressed by one which ends with a sequence of 0s. For instance, $\frac{1}{3}$ can be expressed by the sequence $(1, 0, 0, 0, \ldots)$ or as $(0, 2, 2, 2, \ldots)$. Hence the Cantor set C consists precisely of the points x whose ternary expression does not contain 1s. If $D = \{0, 2\}$ has the discrete topology, then the function $C \longrightarrow \prod_{\mathbb{N}} D$ given by $x \mapsto (x_1, x_2, x_3, \ldots)$ determines a homeomorphism. The set C is not discrete, since in a discrete set the only convergent sequences are those which are eventually constant, while in C the sequence $\{\frac{1}{3^n}\}$ converges to 0.)

8.1.9 EXERCISE. Using the fact that every point in the Cantor set has a unique expression as a sequence $x_1, x_2, x_3, \ldots$, where $x_i = 0, 2$, show that it is an uncountable set. (*Hint:* Emulate the proof that the set of real numbers is uncountable.)

8.1.10 EXERCISE. Let X be a topological space. Show that X is compact if and only if every net $\{x_a\}$ in X has a cluster point. Consequently X is compact if and only if every net in X has a convergent subnet. (See 7.6.18.)

8.1.11 **Proposition**. *Let X be a topological space and take a subset $A \subset X$. Then A, with the relative topology, is compact if and only if every open cover $\{U_\lambda\}$ of A in X contains a finite subcover.*

Proof: Let A be compact as a space with the relative topology and let $\{U_\lambda\}$ be an open cover of A in X. $\{U_\lambda \cap A\}$ is a cover of A by sets which are open in A. Since A is compact, there is a finite number of open sets $U_1 \cap A, \ldots, U_k \cap A$ in this cover such that $A = U_1 \cap A \cup \cdots \cup U_k \cap A$. Hence $A \subset U_1 \cup \cdots \cup U_k$.

Conversely let $\{V_\lambda\}$ be a family of open sets of A such that $A = \bigcup V_\lambda$. Since A has the relative topology, there are open sets U_λ in X such that $V_\lambda = U_\lambda \cap A$. Therefore $A \subset \bigcup U_\lambda$. By assumption, we can take a finite number of such open sets $U_1, \ldots, U_k$ such that $A \subset U_1 \cup \cdots \cup U_k$. Consequently $A = V_1 \cup \cdots \cup V_k$. □

The usual way of defining the concept of compact set is as in the statement of the previous proposition. This way it might seem that compactness would be a property that depends of the way in which the set is embedded in the space. However the statement of the proposition says that the compactness concept is inherent to the set seen as a topological space, rather than to the way it lies inside the larger space.

8.1.12 REMARK. Not every subspace of a compact space is compact. For example, using the Heine–Borel–Lebesgue theorem for $\mathbb{R}$, $(0,1)$ is not compact. However $[0,1]$ is compact. To see directly that the open interval $(0,1)$ is not compact, consider the open cover $\{(\frac{1}{n}, 1 - \frac{1}{n})\}$, $n \in \mathbb{N}$. This cover does not contain a finite subcover.

8.1.13 **Theorem**. *Let X be a compact topological space and take $A \subset X$. If A is closed, then A is compact.*

Proof: Let $\{F_\lambda\}$ be a family of closed sets in A whose intersection is empty. Since A is closed in X, the sets F_λ are closed in X for all λ. Since X is compact, there exists a finite subfamily of the given family, whose intersection is empty. Hence A is compact. □

8.1.14 EXERCISE. Provide proofs of Theorem 8.1.13 using other conditions for compactness.

8.1.15 **Theorem**. *Let X be a Hausdorff space and take $A \subset X$. If A is compact, then A is closed.*

Proof: Assume $A \neq \emptyset$ and take $x \in \overline{A}$. The neighborhood filter $\mathcal{N}_x$ of x in X induces a filter $\mathcal{F}$ in A and since A is compact, $\mathcal{F}$ has a cluster point $a \in A$. This point a is also a cluster point of the continuation $\mathcal{F}' = i(\mathcal{F}) = ii^{-1}(\mathcal{N}_x)$ of $\mathcal{F}$ to X, which is finer than $\mathcal{N}_x$, where $i : A \hookrightarrow X$ is the inclusion map. Hence $\mathcal{F}' \to x$ and since X is a Hausdorff space, x is the only cluster point of $\mathcal{F}'$. Thus $x = a$. Therefore $x \in A$ and so A is closed in X. □

Theorem 8.1.15 and Proposition 8.1.13 can be written together as follows.

8.1.16 **Corollary**. *Let X be a compact Hausdorff space and take $A \subset X$. Then A is compact if and only if A is closed.* □

8.1.17 EXERCISE. Let X be a Hausdorff space and take a compact subset $A \subset X$. If $x \in X - A$, show that there are disjoint open sets U and V in X such that $x \in U$ and $A \subset V$. Observe that this assertion generalizes 8.1.15.

8.1.18 **Theorem**. *Let X be a topological space. Every finite union $\bigcup_{i=1}^{n} A_i$ of compact sets A_i in X is a compact set.*

Proof: Take open cover $\mathcal{U} = \{U_j\}_{j \in \mathcal{J}}$ of $\bigcup_{i=1}^{n} A_i$. Then $\mathcal{U}$ is an open cover of A_i for all $i = 1, \dots, n$. Since each A_i is compact, there are $U_{i,1}, \dots, U_{i,m_i} \in \mathcal{U}$ that cover A_i. Hence the finite family

$$U_{1,1}, \dots, U_{1,m_1}, \dots, U_{n,1}, \dots, U_{n,m_n} \in \mathcal{U}$$

covers $\bigcup_{i=1}^{n} A_i$. □

8.1.19 **Theorem**. *Let X be a Hausdorff space. Every nonempty intersection $\bigcap_{i \in \mathcal{I}} A_i$ of compact sets A_i in X is a compact set.*

Proof: Since X is a Hausdorff space, each A_i is closed and thus $\bigcap_{i \in \mathcal{I}} A_i$ is closed. Since this intersection is also a closed subset of each A_i, which is compact, then the intersection is compact. □

There are spaces X that are not Hausdorff spaces, where we can find compact sets whose intersection is not compact. Namely, we have the following examples.

8.1.20 Example. Take $X = \mathbb{N}$ with the topology $\mathcal{A} = \{\emptyset, \mathbb{N}, A \mid A \subset 2\mathbb{N}\}$. The set $C_k = \{k\} \cup 2\mathbb{N}$ is compact if k is odd. However $C_k \cap C_l = 2\mathbb{N}$, where k, l are odd, $k \neq l$, is not compact, since it is discrete and infinite.

8.1.21 Example. Let $Y = \{0, 1\}$ be furnished with the indiscrete topology and take $X = \mathbb{R} \times Y$ with the product topology. The sets

$$A = (0,3) \times \{0\} \cup \{0,3\} \times \{1\} \quad \text{y} \quad B = (1,2) \times \{0\} \cup \{1,2\} \times \{1\}$$

are compact. For any $a < b \in \mathbb{R}$, the map

$$q : [a,b] \longrightarrow (a,b) \times \{0\} \cup \{a,b\} \times \{1\},$$

given by $q(x) = (x, 0)$ if $x \neq a, b$ and by $q(a) = (a, 1)$, $q(b) = (b, 1)$, is continuous and surjective. Therefore, by 8.1.22 below, since the interval $[a, b] \subset \mathbb{R}$ is compact, $(a, b) \times \{0\} \cup \{a, b\} \times \{1\}$ is also compact. Hence A and B ar compact. However, we have that the intersection $A \cap B = (1, 2) \times \{0\}$ is not compact.

The following result provides us with one of the most important properties of continuity related to compactness.

8.1.22 **Theorem**. *Let X be a compact space and let $f : X \longrightarrow Y$ be a continuous map. Then the image of X under f, $f(X)$, is compact.*

Proof: Let $\mathcal{C}$ be an open cover of $f(X)$ in Y and consider the family $f^{-1}(\mathcal{C}) = \{f^{-1}(U) \mid U \in \mathcal{C}\}$. Since f is continuous, $f^{-1}(\mathcal{C})$ is an open cover of X. Since X is compact, there are finitely many open sets $U_1, \ldots, U_k$ such that $X = f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_k)$. Therefore $f(X) \subset U_1 \cup \cdots \cup U_k$. □

8.1.23 Exercise. Let X be a nonempty compact Hausdorff space and let $f : X \longrightarrow X$ be continuous. Prove that there exists a nonempty closed set $A \subset X$ such that $f(A) = A$.

8.1.24 Definition. Let X be a topological space and take $A \subset X$. We say that A is *relatively compact* if the closure $\overline{A}$ is compact.

8.1.25 Examples.

(a) By the Heine–Borel–Lebesgue Theorem, any bounded set in a Euclidean space is relatively compact. Conversely any relatively compact set in a Euclidean space is bounded.

(b) If X is a compact space, then every subset of X is relatively compact.

The next example due to Raúl Pérez shows that not every compact set is relatively compact. Of course, this can only happen inside a non-Hausdorff space.

8.1.26 Example. Take the set of irrational numbers of the unit interval together with 1, namely $X = I \cap (\mathbb{R} - \mathbb{Q}) \cup \{1\}$ and take the surjective map $q : (0,1] \longrightarrow X$ given by

$$q(x) = \begin{cases} x & \text{if } x \in I \cap (\mathbb{R} - \mathbb{Q}) \\ 1 & \text{if } x \in I \cap \mathbb{Q}\,. \end{cases}$$

Give X the identification topology. The topological space X is neither Hausdorff nor compact. To see this take a sequence $1 > x_1 > x_2 > x_3 > \cdots$ of positive irrationals that tends to zero (in the usual topology). Then the family $\{U_n\}$ given by $U_n = q(\bigcup_{k \geq n}(x_{k+1}, x_k) \cup (x_k, 1]) \subset X$ is an open cover of X that does not contain a finite subcover.

Namely, take $0 < a < b < 1$ and $A = q[a,b] \subset X$. Since $[a,b]$ is compact (Heine–Borel Theorem) and q is continuous, then A is a compact set. However its closure $\overline{A}$ coincides with X, which is not compact. To see this take any nonempty open subset U of X that contains 1 as an element and since $q^{-1}(U)$ is an open subset that contains $I \cap \mathbb{Q}$, then it also contains irrational numbers in any interval, in particular in $[a,b]$. Thus $A \cap U \neq \emptyset$. Therefore $\overline{A}$ is not compact, i.e. A is not relatively compact.

The previous example shows the convenience of working with Hausdorff spaces, as many authors assume of their compact spaces. In particular, if X is Hausdorff, then every compact set $A \subset X$ is relatively compact, since it is already closed (because X is Hausdorff), and so $\overline{A} = A$. By the way, that example also shows that identification spaces, even of the nicest spaces such as an interval, can be quite nasty.

Assume now that A is a relatively compact subset of a topological space X. Therefore any filter in A is such that its extension to the closure $\overline{A}$ has a cluster point. Therefore its extension to all of X has also a cluster point, since a filter $\mathcal{F}$ in X is the extension of a filter in A if and only if $A \in \mathcal{F}$. Then we have the following.

8.1.27 **Theorem**. *Let X be a topological space and let $A \subset X$ be relatively compact. If $\mathcal{F}$ is a filter in X that contains A, then $\mathcal{F}$ has a cluster point.* □

In what follows, we shall prepare the proof of the Heine–Borel–Lebesgue Theorem.

8.1.28 DEFINITION. If X is a metric space, we say that a subset $A \subset X$ is *bounded* if $A \subset B_n(x)$ for some point $x \in X$ and some natural number n.

8.1.29 EXERCISE. Prove that if X is a metric space, then a subset $A \subset X$ is bounded if and only if for any point $x \in X$ there exists $n \in \mathbb{N}$ such that $A \subset B_n(x)$.

8.1.30 **Lemma**. *Every compact set A in a metric space X is closed and bounded.*

Proof: Since X is Hausdorff, then A is closed. Given any point $x_0 \in X$, the family of open balls $\{B_n^\circ(x_0) \mid n \in \mathbb{N}\}$ is clearly an open cover of X and, in particular, of A. Since A is compact, we may find a finite subcover. Therefore $A \subset B_n^\circ(x_0)$ for some $n \in \mathbb{N}$, since each open ball in the cover is contained in the next one with respect to the order of the naturals. Consequently A is bounded. □

8.1.31 NOTE. Remember (1.3.6) that given a metric space X with metric d, we may find a bounded metric d' such that it determines the same topology on X. Therefore, we have that any metrizable space admits a bounded metric and therein, every set, including X itself, is bounded. This shows that the inverse of Lemma 8.1.30 is false in general.

If X is a compact space and $f : X \longrightarrow \mathbb{R}$ is continuous, then Theorem 8.1.22 claims that the image of $f(X) \subset \mathbb{R}$ is a compact set, and by the previous lemma, it is closed and bounded. We have therefore the following consequence.

8.1.32 **Corollary**. *Any real continuous function $f : X \longrightarrow \mathbb{R}$ defined on a compact space X reaches its maximum and its minimum. In other words, there are points $x_0, x_1 \in X$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in X$.* □

8.1.33 EXERCISE. Show that every metric d in a compact space X is bounded, i.e. there exists $K > 0$ such that $d(x, y) \leq K$ for all $x, y \in X$.

8.1.34 **Theorem**. *Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be continuous. If X is compact and Y is Hausdorff, then f is a closed map. Moreover, if f is surjective, then f an identification.*

Proof: Let $A \subset X$ be closed. Since X is compact, A is compact, and since f is continuous, $f(A)$ is compact. Moreover, since Y is Hausdorff, $f(A)$ is closed. Therfore f is closed.

If we now also assume that f is surjective and $B \subset Y$ is such that $f^{-1}(B)$ is closed, then, since f closed, the image $B = ff^{-1}(B)$ is closed. This proves that f is an identification. □

The following is a very useful result, which clearly follows from the previous theorem.

8.1.35 **Corollary**. *Let X be a compact space and Y a Hausdorff space, and let $f : X \longrightarrow Y$ be a continuous bijective map. Then f is a homeomorphism.* □

8.1.36 Exercise. Using geometric arguments, and not necessarily explicit formulas, show that there are identifications $\mathbb{B}^2 \longrightarrow \mathbb{S}^1$, $\mathbb{S}^2 \longrightarrow \mathbb{B}^2$, $\mathbb{S}^2 \longrightarrow \mathbb{S}^1$, $\mathbb{S}^2 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$, and $\mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^2$.

8.1.37 Remark. The combination of the properties of being compact and Hausdorff is particularly interesting. Namely, let X be a compact Hausdorff space. If X' denotes another space with the same underlying set and a coarser topology, then the identity map $f = \mathrm{id} : X \longrightarrow X'$ is continuous. Hence, if $A \subset X$ is closed and thus compact, then $f(A) \subset X'$ is compact. If X' were Hausdorff, then $f(A)$ would be closed, so that f would be a closed map. Then the topology of X' would be finer than that of X and so $X = X'$. We have shown that the topology of X is the coarsest that, being compact, is Hausdorff.

Conversely, if now X' were again a topological space with the same underlying set as X, but with a finer topology, el espacio con el then $f = \mathrm{id} : X' \longrightarrow X$ would be continuous. Therefore, f would be a bijective map from a compact space to a Hausdorff space. Thus it would be a homeomorphism. Again the topology of X would be the same as that of X'. This shows that the topology of X is the finest that, being Hausdorff, is compact.

The previous remark proves the next.

8.1.38 **Theorem**. *The topology of a compact Hausdorff space X is the finest that makes it compact and the coarsest that makes it Hausdorff. This makes the topology of a compact Hausdorff space in some sense unique.* □

8.1.39 Definition. We say that a topological space X has the *maximal compact* if X is compact and with any strictly finer topology it is not compact any more.

The converse of Theorem 8.1.38 is not true, that is, a minimal Hausdorff topology, does not have to be compact, nor a maximal compact topology has to be Hausdorff. To see an example, the one may read [17, Examples 99 and 100].

8.1.40 Exercise. Show that a space X has the maximal compact topology if and only if every compact set in X is closed.

8.1.41 Exercise. Show that every space X with the maximal compact topology is T_1. (Compare this claim with 8.1.38.)

The next result presents the behavior of compactness with respect to the topological product.

8.1.42 **Theorem**. (Tychonoff) *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a nonempty family of nonempty topological spaces and consider the topological product $X = \Pi_{\lambda\in\Lambda} X_\lambda$. Then X is compact if and only if X_λ is for each $\lambda \in \Lambda$.*

Proof: Since the projection $p_\lambda : X \longrightarrow X_\lambda$ is continuous and surjective, then, if X is compact, so is X_λ is compact for each $\lambda \in \Lambda$.

Conversely, let X_λ be compact for each $\lambda \in \Lambda$. Let $\mathcal{U}$ be an ultrafilter in X. By Proposition 7.5.5, the image of $\mathcal{U}$ under the projection, $p_\lambda(\mathcal{U})$, is an ultrafilter in X_λ and since X_λ is compact, $p_\lambda(\mathcal{U})$ converges. Consequently, by 7.5.1, $\mathcal{U}$ converges and therefore X is compact. □

By the Tychonoff Theorem 7.5.4 we have the following result.

8.1.43 **Corollary**. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a nonempty family of nonempty topological sets and take their topological product $X = \Pi_{\lambda\in\Lambda} X_\lambda$. Then X is a compact Hausdorff space if and only if X_λ is a compact Hausdorff space for each $\lambda \in \Lambda$.* □

Exercise 8.1.8 provides an example of an infinite product of discrete spaces which is not discrete, although it is compact. Since this product has an infinite underlying Compact.sets in a topological spaceset, it is impossible that it is discrete, because it would not be compact.

Compact sets in a topological space have in many aspects a similar behavior to points. The next results show this behavior. In finite products we have the following.

8.1.44 **Theorem**. *Let X and Y be topological spaces. If $A \subset X$ and $B \subset Y$ are compact and W is a neighborhood of $A\times B$ in $X\times Y$, Then there are neighborhoods U of A in X and V of B in Y such that $U \times V \subset W$.*

Proof: For every point $(x, y) \in A \times B$ there are open neighborhoods $\widehat{U}$ of x in X and $\widehat{V}$ of y in Y such that $\widehat{U} \times \widehat{V} \subset W$. For a fixed $x \in A$, since B is compact, there are finitely many of these neighborhoods, say $\widehat{U}_1, \widehat{U}_2, \ldots, \widehat{U}_k$ de x, as well as corresponding neighborhoods $\widehat{V}_1, \widehat{V}_2, \ldots, \widehat{V}_k$ such that $B \subset \widehat{V}' = \bigcup_{i=1}^k \widehat{V}_i$.

If we write $\widehat{U}' = \bigcap_{i=1}^{k} \widehat{U}_i$, then $\widehat{U}'$ is an open neighborhood of x and $\widehat{V}'$ is an open neighborhood of B such that $\widehat{U}' \times \widehat{V}' \subset W$. Since A is compact, there are finitely many of these other neighborhoods $\widehat{U}'_1, \widehat{U}'_2, \ldots, \widehat{U}'_l$ as well as corresponding $\widehat{V}'_1, \widehat{V}'_2, \ldots, \widehat{V}'_l$ such that $\widehat{V}'_j$ is an open neighborhood of B, $\widehat{U}'_j \times \widehat{V}'_j \subset W$ and $A \subset U = \bigcup_{j=1}^{l} \widehat{U}'_j$. Then U and $V = \bigcap_{j=1}^{l} \widehat{V}'_j$ re open neighborhoods of A and B, respectively, such that $U \times V \subset W$. □

8.1.45 **Theorem**. *Let X be a Hausdorff space. If A, $B \subset X$ are disjoint compact sets, then there exist disjoint open sets U, $V \subset X$ such that $A \subset U$ and $B \subset V$.*

Proof: Let $\Delta \subset X \times X$ be the diagonal. Since X is Hausdorff, by 7.1.30, Δ is closed. Since A and B are disjoint, then $A \times B \subset W = X \times X - \Delta$, and W is an open neighborhood of $A \times B$. Hence, by 8.1.44, there are open sets U, $V \subset X$ such that $A \times B \subset U \times V \subset W$. Hence $A \subset U$, $B \subset V$ and U and V are disjoint (since their product does not meet the diagonal). □

The following is an immediate consequence.

8.1.46 **Corollary**. *Let X be a compact Hausdorff space. If A, $B \subset X$ are disjoint closed sets, then there exist disjoint open sets U, $V \subset X$ such that $A \subset U$ and $B \subset V$.* □

The separability property for closed sets stated in the previous corollary is known as *normality.* In other words, *every compact Hausdorff space is normal.* This property will be analyzed in detail in the next chapter (9.1).

8.1.47 Exercise. Let X be a Hausdorff space. Show that a finite union of compact subsets of X is compact.

Next exercise shows the relationship between compactness and topological sums.

8.1.48 Exercise. Show that a topological sum $X = \coprod_{\lambda \in \Lambda} X_\lambda$ is a compact space if and only if Λ is finite and each X_λ is compact.

8.1.49 Exercise. Let X be a compact Hausdorff space. Show that for every point $x \in X$ the connected component C_x, that contains x, coincides with $\bigcap\{C \subset X \mid C \text{ es abierto y cerrado y } C \supset C_x\}$. (*Hint:* Use 8.1.45.)

8.1.50 Exercise. Let X be a connected, compact and Hausdorff space. Prove that for every set $A \subset X$ there exists a compact connected set C containing A, which is minimal, i.e. such that if D is a compact connected set and $A \subset D \subset C$, then $D = C$. (*Hint:* Use Zorn's lemma.)

8.1.51 Exercise. Show that every locally connected compact set has finitely many connected components. (*Hint:* Use Exercise 6.2.3.) What happens if X is not locally connected? (*Hint:* Analyze the Cantor set. See 2.2.7.)

8.1.52 Exercise. Let X be a compact metric space and let $f : X \longrightarrow X$ be an isometry, i.e. a map such that $d(f(x), f(y)) = d(x, y)$. Prove that f is surjective. (*Hint:* If one takes $y \in X$, then the sequence y, $f(y)$, $f(f(y)), \ldots$ has points as close to y as desired.)

To finish this section, we shall give an interesting characterization of compact spaces. For that we require the following.

8.1.53 Definition. Let $f : X \longrightarrow Y$ be a continuous map. Consider the following property of f:

(F) For every filter $\mathcal{F}$ en X and for every cluster point $y \in Y$ of the image filter $f(\mathcal{F})$ there exists a cluster point $x \in X$ of $\mathcal{F}$ such that $f(x) = y$.

In this case we simply say that f *has property* (F).

8.1.54 **Lemma**. *Let $f : X \longrightarrow Y$ be a continuous map. If f has property* (F), *then f is closed.*

Proof: Let $A \subset X$ be a nonempty closed set and $y \in \overline{f(A)}$. If $\mathcal{F}_A$ denotes the filter of all supersets of A in X, then its image $f(\mathcal{F}_A)$ is the filter of the supersets of $f(A)$ in Y. Since $y \in \overline{f(A)} \subset \overline{G}$ for any $G \in f(\mathcal{F}_A)$, y is a cluster point of $f(\mathcal{F}_A)$. By property(F), there exists a cluster point $x \in X$ of $\mathcal{F}_A$ such that $f(x) = y$. Hence $x \in \overline{A} = A$ and $y = f(x) \in f(A)$, which shows that $f(A)$ is closed. □.

8.1.55 **Lemma**. *Let $f_\lambda : X_\lambda \longrightarrow Y_\lambda$, $\lambda \in \Lambda$, be a family of maps with property* (F), *then the product map $f = \prod f_\lambda : X = \prod X_\lambda \longrightarrow Y = \prod Y_\lambda$ also has property* (F). *Therefore it is a closed map.*

Proof: Assume that each f_λ has property (F). Let $\mathcal{F}$ be a filter in X and $y = (y_\lambda) \in Y$ a cluster point of $f(\mathcal{F})$. Since the projection $q_\lambda : Y \longrightarrow Y_\lambda$ is continuous, then

y_λ is a cluster point of $q_\lambda f(\mathcal{F}) = f_\lambda p_\lambda(\mathcal{F})$, where $p_\lambda : X \longrightarrow X_\lambda$ is the projection. Since f_λ has property (F), the image $p_\lambda(\mathcal{F})$ has a cluster point $x_\lambda \in X_\lambda$ such that $f_\lambda(x_\lambda) = y_\lambda$. By 7.5.6(c), $x = (x_\lambda) \in X$, which is such that $f(x) = y$, is a cluster point of $\mathcal{F}$. Thus f has property (F). □

8.1.56 **Theorem**. *Let X be a topological space. The X is compact if and only if for every space Z the projection* $\operatorname{proj}_Z : X \times Z \longrightarrow Z$ *is a closed map.*

Proof: Assume first that X is compact. If P denotes the singular space, then clearly the only map $X \longrightarrow P$ has property (F). It is also clear that $\mathrm{id}_Z : Z \longrightarrow Z$ has property (F), so that by 8.1.55, also the product of both maps, namely, the projection $\operatorname{proj}_Z : X \times Z \longrightarrow Z$ has property (F). Thus it is a closed map.

Conversely assume that $\operatorname{proj}_Z : X \times Z \longrightarrow Z$ is a closed map for any space Z. Let $\mathcal{F}$ be a filter in X and let $X^*_{\mathcal{F}}$ be as in 7.1.44 (constructed for the *set* X). Take $\Delta = \{(x,x) \mid x \in X\} \subset X \times X^*_{\mathcal{F}}$ and put $F = \overline{\Delta} \subset X \times X^*_{\mathcal{F}}$. Since by assumption $\operatorname{proj}_{X^*_{\mathcal{F}}} : X \times X^*_{\mathcal{F}} \longrightarrow X^*_{\mathcal{F}}$ is closed, then $\operatorname{proj}_{X^*_{\mathcal{F}}}(F)$ is closed in $X^*_{\mathcal{F}}$. Since $X \subset \operatorname{proj}_{X^*_{\mathcal{F}}}(F)$ and by 7.1.45, X is not closed in $X^*_{\mathcal{F}}$, then $\infty \in \operatorname{proj}_{X^*_{\mathcal{F}}}(F)$, i.e. thete exists $x \in X$ such that $(x, \infty) \in F$. Hence every neighborhood V of x in X and for each element $M \in \mathcal{F}$, the neighborhood $V \times (M \cup \{\infty\}$ of (x, ∞) in $X \times X^*_{\mathcal{F}}$ is such that $V \times (M \cup \{\infty\}) \cap \Delta = (V \times M) \cap \Delta \neq \emptyset$. In other words $V \cap M \neq \emptyset$, so that x is a cluster point of $\mathcal{F}$. Therefore, by 8.1.4(C′′′), X is compact. □

8.1.57 Exercise. Let X a noncompact Hausdorff space and define

$$\mathcal{A} = \{A \subset X \mid X - A \text{ is compact}\} \cup \{\emptyset\}\,.$$

Show that $\mathcal{A}$ is a topology in X, Called *cocompact topology.* How does this topology compare with the original topology of X?

8.2 Compactness and countability

Threr are interesting relationships between compactness and several countability properties. In this section we shall study the property of being compact with respect to the behavior of sequences. We shall prove the Lindelöf theorem on the existence of countable subcovers of a given open cover and we shall use the proven results to finally prove the theorems of Heine–Borel–Lebesgue and Bolzano–Weierstrass.

8.2.1 DEFINITION. A topological space X is said to be *countably compact* if every sequence in X has a cluster point. The space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.

We have the following.

8.2.2 **Proposition**.

(a) *Every compact space X is countably compact.*

(b) *Every sequentially compact space X is countably compact.*

Proof:
(a) This is clear. Namely, since X is a compact space, every filter in X has a cluster point. In particular, every elementary filter in X has a cluster point, i.e. every sequence in X has a cluster point.

(b) This is obvious. □

The converse of (b) is not necessarily true. However one has the following.

8.2.3 **Theorem**. *Every countably compact first-countable space X is sequentially compact.*

Proof: Take a sequence $\{x_n\}$ in X and let x be a cluster point of the sequence. Since X is first-countable, we may take a countable neighborhood basis of x, $U_1 \supset U_2 \supset U_3 \supset \cdots \supset U_n \supset \cdots$. Since x is a cluster point of $\{x_n\}$, then $\{x_n \mid n > m\} \cap U_k \neq \emptyset$ for all k and m. Choose $x_{n_1} \in \{x_n\} \cap U_1$. Inductively, assume that we we have chosen $x_{n_i} \in U_i, n_1 < n_2 < \cdots < n_k$ and then choose $x_{n_{k+1}} \in \{x_n \mid n > n_k\} \cap U_{k+1}$. Hence the sequence $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and clearly $x_{n_k} \to x$. □

8.2.4 **Theorem**. (Lindelöf) *Assume X to be a second-countable space. Then X satisfies the* Lindelöf axiom

(L) *Every open cover of X contains a countable subcover.*

Proof: Let $\{Q_n\}$ be a countable basis for the topology of X and let $\{U_\lambda\}$ be an open cover of X. For each λ one has

$$U_\lambda = \bigcup_{Q_n \subset U_\lambda} Q_n \,.$$

Let $n_1 < n_2 < \cdots$ be such that $Q_{n_i} \subset U_\lambda$ for some λ. Thus

$$\bigcup_{i=1}^{\infty} Q_{n_i} = \bigcup_{\lambda\in\Lambda} U_\lambda = X\,.$$

Now take U_{λ_i} such that $Q_{n_i} \subset U_{\lambda_i}$. Clearly

$$\bigcup_{i=1}^{\infty} U_{\lambda_i} = X\,.$$

Therefore $\{U_{\lambda_i}\}$ is a countable subcover of X. □

8.2.5 Definition. A topological space X that satisfies (L) is called a *Lindelöf space.*

8.2.6 **Lemma**. *A space X is countably compact if and only if every countable open cover of X contains a finite subcover.*

Proof: Let X be countably compact and let $\{U_n\}$ be a countable open cover of X. Assume on the contrary that $\{U_n\}$ does not contain a finite subcover. Thus $V_n = \bigcup_{k=1}^{n} U_k \neq X$ for all n. Take $x_n \in X - V_n$. Since X is countably compact, $\{x_n\}$ has a cluster point $x \in X$. Then $x \in U_m$ for some m. On the other hand $U_m \subset V_m$. Therefore V_m is a neighborhood of x that contains only finitely many terms of the sequence. This is a contradiction to the fact that x is a cluster point of the sequence. Hence $\{U_n\}$ contains a finite subcover.

Conversely, assume that every countable open cover of X contains a finite subcover. Assume that there is a sequence $\{x_n\}$ in X with no cluster point. Hence any point $x \in X$ has a neighborhood V such that $V \cap \{x_n \mid n \geq k\} = \emptyset$ for k sufficiently large. Take the open set $V_k = X - \overline{\{x_n \mid n \geq k\}}$. Clearly $V_1 \subset V_2 \subset \cdots \subset V_k \subset \cdots$. Then $\{V_k\}$ es una countable open cover of X, since for any $x \in X$ there is a neighborhood V of x such that $V \cap \{x_n \mid n \geq k\} = \emptyset$ for some k Hence $x \notin \overline{\{x_n \mid n \geq k\}}$. Thus $x \in V_k$. The countable open cover $\{V_k\}$ does not contain a finite subcover since $V_k \neq X$ for all k. □

8.2.7 Note. Many authors define a countably compact space X as a space such that each countable open cover contains a finite subcover. The previous lemma shows that our definition is equivalent.

8.2.8 **Proposition**. *If X is a countably compact Lindelöf space, then X is compact.*

Proof: Let $\{U_\lambda\}$ be an open cover of X. By the Lindelöf axiom, $\{U_\lambda\}$ contains a countable subcover, and by Lemma 8.2.6, this countable cover contains a finite subcover. □

As a consequence of the Lindelöf Theorem and of 8.2.8 we obtain the following.

8.2.9 **Theorem**. *Let X be a second-countable, countably compact space. Then X is compact.* □

8.2.10 **Theorem**. *Let X be a countably compact metric space. Then X satisfies the following:*

(a) *For every open cover $\{Q_\lambda\}$ of X there exists $\delta > 0$, such that for any point $x \in X$ there is some λ such that the $D_\delta(x) \subset Q_\lambda$. (Such a δ is called* Lebesgue number *of the cover.)*

(b) *For every $\varepsilon > 0$ there exist points $x_1, \ldots, x_k \in X$ such that $X = D_\varepsilon(x_1) \cup \cdots \cup D_\varepsilon(x_k)$.*

(c) *X is compact.*

(d) *X is sequentially compact.*

Proof:

(a) Let $\{Q_\lambda\}$ be an open cover which does not admit a Lebesgue number and let $\delta_n = 1/2^n$. Hence there exists a point x_n such that $D_{\delta_n}(x_n) \not\subset Q_\lambda$ for all λ. Since X is countably compact, the sequence $\{x_n\}$ has a cluster point $x \in X$ and there is a $\kappa \in \Lambda$ such that $x \in Q_\kappa$. But Q_κ is open and so there is $m \in \mathbb{N}$ such that $D_{\delta_m}(x) \subset Q_\kappa$. For any point $y \in D_{\delta_{m+1}}(x)$ one has

$$d(z,y) \leq \frac{1}{2^{m+1}} \Longrightarrow d(z,x) \leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} = \frac{1}{2^m}\,.$$

This means $D_{\delta_{m+1}}(y) \subset D_{\delta_m}(x) \subset Q_\kappa$ and therefore, for all $n \geq m+1$, $x_n \notin D_{\delta_{m+1}}(x)$. This contradicts the fact that x is a cluster point.

(b) If this were false, then there woud be an $\varepsilon > 0$such that X is not covered by finitely many balls of radius ε. Take any point $x_1 \in X$. The ball $D_\varepsilon(x_1)$ does not cover X. Now take a point $x_2 \in X - D_\varepsilon(x_1)$. Assume inductively that we have taken points $x_1, \ldots, x_k$ such that $x_i \notin D_\varepsilon(x_1) \cup \cdots \cup D_\varepsilon(x_{i-1})$, $i = 2, \ldots, k$. The union $\bigcup_{i=1}^k D_\varepsilon(x_i)$ does not cover X. Hence take $x_{k+1} \in X - \bigcup_{i=1}^k D_\varepsilon(x_i)$. The sequence $\{x_n\}$ cluster points since given any point $x \in X$, the ball $D_{\varepsilon/2}(x)$ contains at most one point of the sequence, because, by construction, the distance between two of them is greater than or equal to ε.

(c) Let $\{Q_\lambda\}$ be an open cover of X. By (a), this cover has a Lebesgue number δ. By (b), we know that X is covered by finitely many balls of radius δ, namely $X = \bigcup_{i=1}^{k} D_\delta(x_i)$. Since δ is a Lebesgue number of $\{Q_\lambda\}$, for each x_i there is a λ_i such that $D_\delta(x_i) \subset Q_{\lambda_i}$. Therefore $X = \bigcup_{i=1}^{k} Q_{\lambda_i}$.

(d) This is a consequence of 8.2.3, since every metric space is first-countable.□

The following is a basic result.

8.2.11 **Theorem**. *The unit interval $I = [0,1]$ is compact.*

Proof: By the previous theorem, it is enough to prove that I is countably compact, since it is metric. So take a sequence $\{x_n\}$ in I. For each point $c \in I$ put $S_c = \{n \mid x_n < c\}$ and consider the set $C = \{c \mid S_c \text{ is finite}\}$. Clearly $0 \in C$. Therfore there exists $c_0 = \sup C$. We shall see that c_0 is a cluster point of $\{x_n\}$. Take $\varepsilon > 0$. It is obvious that $S_{c_0-\varepsilon/2}$ is finite, so that $x_n > c_0 - \varepsilon$ for infinitely many values of n.

If $c_0 = 1$, then the ball with center c_0 and radius ε in I contains a tail of the sequence. Assume now that $c_0 < 1$. Since $c_0 = \sup C$, $c_0 + \varepsilon \notin C$. Hence $S_{c_0+\varepsilon}$ is infinite. Consequently, also in this case the ball with center c_0 and radius ε in I contains points x_n for infinitely many values of n. Thus c_0 is a cluster point of $\{x_n\}$ □

As a consequence of 8.1.30, the Tychonoff Theorem and the previous theorem, we can finally prove the two statements at the beginning of the chapter (8.1.1 and 8.1.2).

8.2.12 **Theorem**. (Heine–Borel–Lebesgue) *A subset $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof: If A is compact, then by 8.1.30 it is closed and bounded, since $\mathbb{R}^n$ is metric.

Conversely assume that A is closed and bounded. Since A is bounded, there is an interval $J = [a,b] \subset \mathbb{R}$ such that $A \subset J^n \subset \mathbb{R}^n$. Moreover, there is a homeomorphism $J \approx I$, so that J is compact too. By the Tychonoff Theorem 8.1.42, J^n is compact, and since A is closed in $\mathbb{R}^n$, it is also closed in J^n. But since J^n is compact, A is compact. □

As a consequence of the Heine–Borel–Lebesgue Theorem we obtain the other statement at the beginning of the chapter.

8.2.13 **Theorem**. (Bolzano–Weierstrass) *A subset* $A \subset \mathbb{R}^n$ *is closed and bounded if and only if every sequence in* A *has a cluster point in* A.

Proof: By 8.2.12, A is closed and bounded if and only if A is compact, and by 8.2.10, since A is metric, A is compact if and only if A is countably compact. □

8.2.14 EXERCISE.

(a) Similarly to the proof of 8.2.11, show that any totally ordered set with a maximal and a minimal element, and satisfying the supremum axiom, becomes a countably compact space if it is furnished with the order topology.

(b) Prove that it is, in fact, compact. (*Hint:* Use open cover.)

There is a local version of the compactness concept, which is quite useful, in case that the considered space is not compact.

8.2.15 DEFINITION. A topological space X is *locally compact* if for every point $x \in X$ there is a compact neighborhood $U \in \mathcal{N}_x$.

Many authors define a locally compact space adding the condition that it is also a Hausdorff space. This assumption facilitates the statements of several results. We shall add this condition explicitly each time that we need it.

The following is clear.

8.2.16 **Proposition**. *Every compact space* X *is locally compact.* □

8.2.17 EXAMPLES.

(a) A *topological* n*-manifold* is a Hausdorff second-countable space X such that each of its points has a neighborhood U which is homeomorphic to an open set in $\mathbb{R}^n$. Then every topological n-manifold is a a locally compact space.

(b) Take $\mathbb{R}^\omega = \prod_{i=1}^{\infty} \mathbb{R}_i$, where $\mathbb{R}_i = \mathbb{R}$ and take $x = (x_i) \in \mathbb{R}^\omega$. Then any neighborhood of x contains a neighborhood of the form $\prod Q_i$, where Q_i is neighborhood of x_i and $Q_i = \mathbb{R}$ for almost every index i. By the Tychonoff Theorem, a neighborhood such as this cannot be compact. Hence no other neighborhood of x would be compact and thus $\mathbb{R}^\omega$ is not locally compact.

8.2.18 Definition. A topological space X is *regular* if for any point $x \in X$, the closed neighborhoods of x constitute a neighborhood basis. In other words, X is regular if and only if for every $x \in X$ and every $V \in \mathcal{N}_x$, there $W \in \mathcal{N}_x$ such that W is closed and $W \subset V$.

In the next chapter we shall study this property in more detail (9.1.2).

8.2.19 **Theorem**. *Every locally compact Hausdorff space X is regular.*

Proof: Since every compact set in a Hausdorff space X is closed, it is enough to prove that the compact neighborhoods of a point $x \in X$ build up a neighborhood basis of x, if x is locally compact and Hausdorff. Since X is Hausdorff, the set $\{x\} = \bigcap\{V \in \mathcal{N}_x \mid V \text{ is closed}\}$. The closed neighborhoods of x constitute a filter basis of a filter $\mathcal{F}$ which is coarser than the neighborhood filter $\mathcal{N}_x$ of x. But, on the other hand, $\mathcal{F}$ is finer than $\mathcal{N}_x$, since by definition of a cluster point of a filter, x is the only cluster point of $\mathcal{F}$. Since X is compact, by 8.1.6, $\mathcal{F}$ converges to x. This proves that the closed (compact) neighborhoods of x en X build up a neighborhood basis of x.

If X is not compact, but only locally compact, then each $x \in X$ has a compact neighborhood V. Thus V is closed and so x has a neighborhood basis relative to V whose elements are compact neighborhoods. Since V itself is a neighborhood, then this neighborhood basis in V is also a neighborhood basis in X. □

8.2.20 **Corollary**. *A Hausdorff space X is locally compact if and only if each point of X has a basis of compact neighborhoods.* □

Some authors define a locally compact space X as space such that each of its points has a basis of compact neighborhoods. The latter corollary shows that for Hausdorff spaces their and our definition are equivalent.

The property of a space being locally compact is not inherited by subspaces.

8.2.21 Example. $\mathbb{R}$ is locally compact but $\mathbb{Q} \subset \mathbb{R}$ is not locally compact.

We have though the following statements.

8.2.22 **Proposition**. *Let X be a locally compact space $A \subset X$ a closed subset. Then A is locally compact.*

Proof: Take $x \in A$ and let V be a compact neighborhood of x in X. Then $V \cap A$ is a compact neighborhood x in A. □

8.2.23 **Proposition**. *Let X be a locally compact Hausdorff space and $A \subset X$ an open subset. Then A is locally compact.*

Proof: Since the compact neighborhoods of a point $x \in X$ build up a neighborhood basis in X, those neighborhoods which are contained in A are a neighborhood basis of x in A. $\square$

8.2.24 DEFINITION. Let X be a topological space and take $A \subset X$. We say that A is *locally closed* in X if $A = Q \cap C$, where Q is open and C is closed in X.

8.2.25 EXERCISE. Show that A is locally closed in X if and only if for every $x \in A$ there is a neighborhood V of x in X such that $A \cap V$ is closed in V.

We can put together the statements of Propositions 8.2.22 and 8.2.23, in the case of Hausdorff spaces. Namely, we have the following result.

8.2.26 **Theorem**. *Let X a locally compact Hausdorff space and let $A \subset X$ be locally closed. Then A is locally compact.*

Proof: By definition, $A = Q \cap C$ with Q open and C closed in X. By 8.2.22, C is locally compact and, since A is open in C because it is the intersection of C with an open set Q, by 8.2.23, A is locally compact. $\square$

8.2.27 **Theorem**. *Let $\{X_\lambda\}$ be a nonempty family of nonempty topological spaces. The topological product $X = \prod X_\lambda$ is locally compact if and only if all factors are locally compact and all but a finite number of them are compact.*

Proof: Assume first that X is locally compact. Therefore any point $x = (x_\lambda)$ has a compact neighborhood V that contains another neighborhood Q of the form $Q = \prod Q_\lambda$, with Q_λ neighborhood of x_λ and $Q_\lambda = X_\lambda$ for all but a finite number of indexes λ. If $p_\lambda : X \longrightarrow X_\lambda$ is the projection, then the image $p_\lambda(V)$ is a compact neighborhood of x_λ, since $Q_\lambda \subset p_\lambda(V)$. Hence x_λ, which can be arbitrarily chosen, has a compact neighborhood, i.e. X_λ is locally compact. Moreover, $p_\lambda(V) = X_\lambda$ all but a finite number of indexes λ, hence, for all these indexes, X_λ is compact.

Conversely, assume that X_λ is locally compact for all λ and that X_λ is compact for all but a finite number of indexes λ. Let us say that $\lambda = \kappa_1, \dots, \kappa_n$ are these indexes. Take $x = (x_\lambda) \in X$ and take a compact neighborhood V_{κ_i} of x_{κ_i}. Take $V_\lambda = X_\lambda$ for $\lambda \neq \kappa_i$, $i = 1, \dots, n$. Then it is clear that $V = \prod V_\lambda$ is a compact neighborhood of x in X. $\square$

8.3 The Alexandroff compactification

In this section we shall study the simplest of all compactifications of a space, which is obtained by adding to the space only one point. We start with an example.

8.3.1 Example. We can embed $\mathbb{R}^n$ into the n-sphere $\mathbb{S}^n$ via the map $i : \mathbb{R}^n \longrightarrow \mathbb{S}^n$ given by

$$i(x_1, \ldots, x_n) = \frac{(2x_1, 2x_2, \ldots, 2x_n, x_1^2 + \cdots + x_n^2 - 1)}{x_1^2 + \cdots + x_n^2 + 1} .$$

This map is the inverse of the *stereographic projection*, and it embeds $\mathbb{R}^n$ as the complement of the *north pole* $(0, \ldots, 0, 1)$ of $\mathbb{S}^n$. Namely, $\mathbb{R}^n$ becomes a subspace of $\mathbb{S}^n$ such that its complement is one point (see 2.6.6).

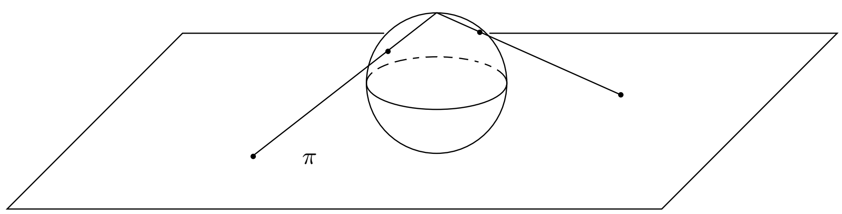


Figure 8.1 The stereographic projection

In the example above we observe that $\mathbb{R}^n$, which is a noncompact space, can be completed to a compact space by just adding one extra point. Indeed we obtain $\mathbb{S}^n$ after adding to $\mathbb{R}^n$ a *point at infinity*. We can easily prove that $\mathbb{R}^n$ seen as a subspace of $\mathbb{S}^n$ is dense.

8.3.2 Definition. Let X be a noncompact topological space. A *compactification* of X is a space X' together with an embedding $i : X \longrightarrow X'$ such that the image $i(X)$ is dense in X'.

Hence Example 8.3.1 shows that the inverse of the stereographic projection $i : \mathbb{R}^n \longrightarrow \mathbb{S}^n$ is a compactification of $\mathbb{R}^n$.

8.3.3 Definition. Let X be a Hausdorff space and consider the set X^* consisting of the union of X with an additional point ∞. Namely $X^* = X \cup \{\infty\}$. Take the topology on X^* given by

$$\mathcal{A}^* = \mathcal{A} \cup \{A \subset X^* \mid \infty \in A, X^* - A \subset X \text{ is compact}\} ,$$

where $\mathcal{A}$ is the original topology on X. The resulting topological space X^* is called the *Alexandroff construction* of X.

8.3.4 EXAMPLE. If X is discrete and infinite and $\mathcal{F}$ is the cofinite filter, built up with the cofinite sets in X, 7.1.10(f), then the Alexandroff construction of X^* coincides with the space $X^*_{\mathcal{F}}$ associated to the filter $\mathcal{F}$ after 7.1.44 (see 8.3.8).

Next we see that $\mathcal{A}^*$ is indeed a topology on X^*.

8.3.5 **Proposition**.

(a) *$\mathcal{A}^*$ is a topology on X^* that induces on $X \subset X^*$ the original topology $\mathcal{A}$.*

(b) *X^* with the topology $\mathcal{A}^*$ is a compact space.*

(c) *X is open in X^*.*

Hence, if X is a noncompact space, then its Alexandroff construction X^*, together with the natural inclusion is a compactification.

Proof:

(a) We have to show that arbitrary unions and finite intersections of elements of $\mathcal{A}^*$ that contain ∞ are again elements of $\mathcal{A}^*$. Let $\{A_\lambda\}$ be a family of elements of $\mathcal{A}^*$ such that $\infty \in A_\lambda$ for all λ. Take $A = \cup A_\lambda$. Then $\infty \in A$ and $X^* - A = \cap(X^* - A_\lambda)$, which is compact since each $X^* - A_\lambda$ is compact too and X is Hausdorff. Then $\bigcup A_\lambda \in \mathcal{A}^*$. Thus $\mathcal{A}^*$ is a topology on X^*.

Take now elements A_1 and A_2 of $\mathcal{A}^*$ such that $\infty \in A_1 \cap A_2$. Then $X^* - (A_1 \cap A_2) = (X^* - A_1) \cup (X^* - A_2)$, which is clearly compact. Therefore $A_1 \cap A_2 \in \mathcal{A}^*$.

(b) Let $\{Q_\lambda\}$ be an open cover of X^*. Take λ_0 such that $\infty \in Q_{\lambda_0}$. The set $C = X^* - Q_{\lambda_0} \subset X$ is compact and $\{Q_\lambda\}$ is an open cover of C in X^*. Hence there is a finite subcover of C, say consisting of $Q_{\lambda_1}, \dots, Q_{\lambda_k}$. One clearly has that the finite family $\{Q_{\lambda_0}, Q_{\lambda_1}, \dots, Q_{\lambda_k}\}$ is a cover X^*. Thus X^* is compact.

(c) This is clear, since $X \in \mathcal{A}$.

If X is noncompact, then the closure $\overline{X} = X^*$, since X and X^* differ by one point and $\overline{X}$ must contain points not in X. In other words, the closure $\overline{X}$ contains $\{\infty\}$. □

8.3.6 DEFINITION. Let X be a noncompact space. The space X^* is called the *Alexandroff compactification* of X. In some texts it is also called *one-point compactification* of X.

The assumption that X is Hausdorff is not essential. In the same way that we proved 8.3.5, one can prove the next result.

8.3.7 **Theorem**. *Let X be a topological space with topology $\mathcal{A}$ and let $X^* = X \cup \{\infty\}$. Define*

$$\mathcal{A}^* = \mathcal{A} \cup \{A \subset X^* \mid \infty \in A, \quad X^* - A \quad \text{is compact and closed}\}\,.$$

Then

(a) *$\mathcal{A}^*$ is a topology in X^* that has X as a subspace*

(b) *X^* with the topology $\mathcal{A}^*$ is compact, and*

(c) *if X is not compact, then it is dense in X^*.*

8.3.8 Exercise. Compare the given construction of X^* with the construction of $X^*_{\mathcal{F}}$ given in 7.1.44, and show that if X is discrete, both resulting spaces coincide.

8.3.9 Exercise. Let X be a compact space. Show that the Alexandroff construction X^* of X, 8.3.3, yields the topological sum of X and a point, i.e.

$$X^* = X \sqcup \{\infty\}\,.$$

Hence when X is compact, X is not dense in X^*.

Even when the space X is Hausdorff, the Alexandroff compactification X^* of X is not necessarily Hausdorff.

8.3.10 Example. Let $\mathbb{Q} \subset \mathbb{R}$ have the relative topology. Then $\mathbb{Q}$ is a Hausdorff space. However, $\mathbb{Q}^*$ is not a Hausdorff space. Namely, it is not possible to separate with open neighborhoods ∞ from any other element $q \in \mathbb{Q}$. To see this, take $q \in \mathbb{Q}$. A typical neighborhood of q is an interval $V = (a,b) \cap \mathbb{Q}$. Let A be a neighborhood of ∞ in $\mathbb{Q}^*$. Therefore $C = \mathbb{Q}^* - A$ is compact. If $V \cap A = \emptyset$, then $V \subset C$. Notwithstanding, this is impossible since there are sequences in V (o rational numbers) which in $\mathbb{R}$ converge to irrational numbers. Hence they do not converge in $\mathbb{Q}$, thus contradicting the compactness of C.

8.3.11 Exercise. Let $\mathbb{Q}'$ be the quotient $\mathbb{R}/\mathbb{R} - \mathbb{Q}$ and take the composite

$$e : \mathbb{Q} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{R} - \mathbb{Q}\,.$$

(a) Show that $e : \mathbb{Q} \longrightarrow \mathbb{Q}'$ is an embedding.

(b) Show that $\mathbb{Q}$ is dense in $\mathbb{Q}'$.

(c) Is $\mathbb{Q}'$ compact?

(d) How do $\mathbb{Q}^*$ and $\mathbb{Q}'$ compare, if $\mathbb{Q}^*$ is as in Example 8.3.10?

There is a way to guarantee that the Alexandroff compactification of a Hausdorff space is again a Hausdorff space. We have the following result.

8.3.12 **Theorem**. (Alexandroff) *X is a Hausdorff locally compact space if and only if its Alexandroff compactification X^* is a Hausdorff compact space.*

Proof: First we prove that if X is Hausdorff locally compact, then X^* is Hausdorff. To see this it is enough to prove that given $x \in X$ there are a neighborhood of x and a neighborhood of ∞ which are disjoint. Since X is locally compact, there is a compact neighborhood V of $x \in X$. Take $W = X^* - V$. Then W is a neighborhood of ∞ in X^* which is obviously disjoint to V.

Conversely, if X^* is Hausdorff, then X is also Hausdorff, because it is a subspace. Moreover, if we take $x \in X$, then there are disjoint neighborhoods V of x and W of ∞ in X^*. We may assume that W is open, so that $X^* - W$ is closed and compact in X. Since $V \subset X^* - W$, then $\overline{V} \subset X^* - W$ and thus $\overline{V}$ is a compact neighborhood of x in X, and so X is locally compact. □

8.3.13 Remark. Since X is open in X^*, if X^* is first-countable, then X is also first-countable. Moreover, ∞ has a countable neighborhood basis. On the other hand, if X^* is second-countable, then the open sets of a countable basis of the topology of X^* that contain ∞, form a countable neighborhood basis of ∞. This brings us to the next definition.

8.3.14 Definition. A toplogical space X is *countable at infinity* if $\infty \in X^*$ has a countable neighborhood basis.

8.3.15 **Proposition**. *The Alexandroff construction X^* is first-, resp. second-countable, if and only if X is first-, resp. second-countable and it is countable at infinity.* □

Now we shall study the meaning of the countability at infinity. Let X Hausdorff locally compact space and let $\mathcal{V}$ be a open-neighborhood basis of ∞ in X^*. The complements of the neighborhoods in $\mathcal{V}$ are compact sets in X and since X^* is Hausdorff, then $\{\infty\} = \bigcap_{V \in \mathcal{V}} V$. Hence $X = X^* - \{\infty\} = X^* - \bigcap_{V \in \mathcal{V}} V = \bigcup_{V \in \mathcal{V}} (X^* - V)$, i.e. X is a union of compact sets, one for each element of $\mathcal{V}$. Hence if $\mathcal{V}$ is countable, i.e. if X is countable at infinity, then we have that X is a countable union of compact sets.

Conversely, assume that $X = \bigcup_{n=1}^{\infty} K_n$, where K_n is a compact set for each n. Therefore $\{X^* - K_n\}$ is a countable family of open neighborhoods of ∞ in X^*. We shall use these neighborhoods to construct a countable neighborhood basis of ∞. Since by 8.2.19 every Hausdorff (locally) compact space is regular, there is a compact neighborhood V_1 of ∞ in X^*, such that $V_1 \subset X^* - K_1$. Now we have that $(X^* - K_2) \cap V_1^\circ$ is an open neighborhood of ∞ in X^* too. For the same reason as before, this neighborhood contains a compact neighborhood V_2 of ∞. Inductively assume that we have already constructed a neighborhood V_n of ∞ such that $V_n \subset (X^* - K_n) \cap V_{n-1}^\circ$. Using the same argument as before, there is an open neighborhood V_{n+1} of ∞ such that $V_{n+1} \subset (X^* - K_{n+1}) \cap V_n^\circ$. Thus $V_{n+1} \subset V_n^\circ$ for all n. We now see that $\{V_n\}$ is a neighborhood basis of ∞. For this, let Q be an open neighborhood of ∞. Hence $X^* - Q$ is compact. On the other hand we have

$$\begin{aligned}\bigcup_{n=1}^{\infty}(X^* - V_n) &= X^* - \bigcap_{n=1}^{\infty} V_n \\ &\supset X^* - \bigcap_{n=1}^{\infty}(X^* - K_n) \\ &= X^* - (X^* - \bigcup_{n=1}^{\infty} K_n) \\ &= \bigcup_{n=1}^{\infty} K_n = X\,.\end{aligned}$$

Hence $\{X^* - V_n\}$ is an open cover of the compact set $X^* - Q$. Besides, since $X^* - V_1 \subset X^* - V_2 \subset \cdots$, we have $X^* - Q \subset X^* - V_n$ for n large enough, i.e. $V_n \subset Q$ for some n. This proves that $\{V_n\}$ is a neighborhood basis of ∞.

In fact, we have proved that $X = \bigcup_{n=1}^{\infty} \overline{Q_n}$. Thus we have the following.

8.3.16 **Theorem**. *Let X be a locally compact Hausdorff space, then the following ar equivalent:*

(a) *X is countable at infinity.*

(b) *X is a countable union of compacto sets.*

(c) *$X = \bigcup_{n=1}^{\infty} Q_n$, where Q_n is open for all n, $\overline{Q}_n$ is compact, and $\overline{Q}_n \subset Q_{n+1}$, $n \in \mathbb{N}$.* □

8.3.17 Note. Notice that the fact that X is a countable union of compact sets does not imply that X is locally compact. For example, consider $X = \mathbb{Q}$.

Now we analyze an important example, where several of the concepts introduced in this chapter play a role.

8.3.18 EXAMPLE. The Alexandroff compactification $\mathbb{C}^*$ of the complex plane $\mathbb{C}$, which by 8.3.1 is homeomorphic to the 2-sphere, is the so-called *Riemann sphere*. If we now take the 3-sphere $\mathbb{S}^3 \subset \mathbb{C}^2 \approx \mathbb{R}^4$, defined by

$$\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 \mid \|z\|^2 + \|w\|^2 = 1\}\,,$$

there is a continuous surjective map

$$p : \mathbb{S}^3 \longrightarrow \mathbb{C}^*, \qquad p(z,w) = \begin{cases} \frac{z}{w} & \text{if } w \neq 0 \\ \infty & \text{if } w = 0\,. \end{cases}$$

Since $\mathbb{S}^3$ is compact and $\mathbb{C}^*$ is Hausdorff, then p is an identification.

Define the *complex projective space* $\mathbb{CP}^n$ as the quotient space $\mathbb{S}^{2n+1}/\!\sim$, where $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$. Hence, if $x = (x_1, \dots, x_{n+1})$, $y = (y_1, \dots, y_{n+1}) \in \mathbb{S}^{2n+1}$, $x = \lambda y$ for $\lambda \in \mathbb{C}$, we can take the identification $q : \mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n$ (compare with 4.2.17(d)). Then, in particular, the map $p : \mathbb{S}^3 \longrightarrow \mathbb{C}^*$ defined above is compatible with the identification $q : \mathbb{S}^3 \longrightarrow \mathbb{CP}^1$ and hence we obtain a homeomorphism $\varphi : \mathbb{CP}^1 \longrightarrow \mathbb{C}^* \approx \mathbb{S}^2$.

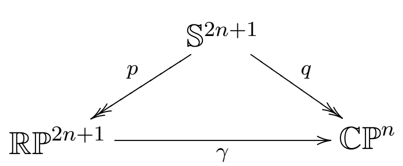
When we take $\mathbb{CP}^n$ as a quotient of $\mathbb{S}^{2n+1}$, what we in fact are doing is taking an arbitrary point $y \in \mathbb{S}^{2n+1}$ and the *orbit* $\{\lambda y \mid \lambda \in \mathbb{S}^1\} \subset \mathbb{S}^{2n+1}$, and identifying each of these orbits in one point. (In other words, we have an action of the group $\mathbb{S}^1$ on the $(2n+1)$-sphere and $\mathbb{CP}^n$ is the orbit space of this action, as we already explained in 5.5.1-5.5.5. The only difference is that there we considered the action of finite groups and here we are dealing with the action of the group $\mathbb{S}^1$ which has a topological structure involved in the action.)

Concluding we may say that there is a map (identification) $\eta : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$, such that its *fiber*, i.e. the inverse image of each point, is a copy of $\mathbb{S}^1$, which plays an important role in homotopy theory and is known as the *Hopf fibration*.

8.3.19 EXERCISE. In the previous example, prove that all assertions that we made hold.

8.3.20 EXERCISE. Prove that the map $\varphi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$ such that $\varphi(z,w) = (2z\overline{w}, \|z\|^2 - \|w\|^2)$ coincides with the Hopf fibration η defined above.

8.3.21 EXERCISE. Prove that there is a canonical map $\gamma : \mathbb{RP}^{2n+1} \longrightarrow \mathbb{CP}^n$, so that the diagram



commutes, where p and q are the canonical identifications. Show that φ is surjective and that its fiber $\gamma^{-1}(x) \approx \mathbb{S}^1$. More precisely, notice that there is an action of $\mathbb{S}^1$ on $\mathbb{RP}^{2n+1}$, i.e. a way of multiplying the elements of $\mathbb{RP}^{2n+1}$ by unit complex numbers, in such a way that the corresponding orbit space is $\mathbb{CP}^n$. (*Hint:* For the latter statement, consider the action $\zeta^2 \cdot [z] = [\zeta z]$, where $\zeta \in \mathbb{S}^1$, $z \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$ and $[z] = p(z) \in \mathbb{RP}^{2n+1}$. Notice that ζ^2 is any element of $\mathbb{S}^1$ that determines ζ up to sign.)

8.3.22 Exercise. Prove that the map $\alpha : \mathbb{S}^1 \times \mathbb{RP}^{2n+1} \longrightarrow \mathbb{RP}^{2n+1}$ given by $\alpha(\zeta^2, [z]) = [\zeta z]$ is well defined and is such that for any given point $z \in \mathbb{RP}^{2n+1}$, the map $\mathbb{S}^1 \longrightarrow \mathbb{RP}^{2n+1}$ given by $\zeta^2 \mapsto [\zeta z]$ is an embedding. Moreover prove that if we identify two points $[z]$ and $[z']$ in $\mathbb{RP}^{2n+1}$ if and only if $[z'] = [\zeta z]$, then the quotient map is (homeomorphic to) $\mathbb{CP}^n$.

To close this section, we analyze an interesting example.

8.3.23 Example. As we have already mentioned in 5.4.18, the projective plane $\mathbb{RP}^2$ can be obtained from the disk $\mathbb{B}^2$ if one identifies antipodal points on the boundary, namely

$$\mathbb{RP}^2 = \mathbb{B}^2/\!\sim \ \text{ where } x \sim y \Leftrightarrow x, y \in \mathbb{S}^1 = \partial\mathbb{B}^2 \text{ and } x = \pm y\,.$$

If $N = (0,1)$, $S = (0,-1) \in \mathbb{S}^1 \subset \mathbb{B}^2$ are the poles, then we have a homeomorphism

$$\varphi : \mathbb{B}^2 - \{N, S\} \longrightarrow [-1,1] \times (-1,1)\,,$$

given by

$$\varphi(x_1, x_2) = \Big(\frac{x_1}{\sqrt{1 - x_2^2}}, x_2\Big)\,,$$

with inverse

$$\psi : [-1,1] \times (-1,1) \longrightarrow \mathbb{B}^2 - \{N, S\}\,,$$

given by

$$\psi(s,t) = (s\sqrt{1-t^2}, t)\,.$$

If $q : \mathbb{B}^2 \longrightarrow \mathbb{RP}^2$ denotes the identification described above and $P = q(N) = q(S)$, then the homeomorphism φ determines a homeomorphism in in the quotient spaces

$$\overline{\varphi} : \mathbb{RP}^2 - \{P\} = q(\mathbb{B}^2 - \{N, S\}) \longrightarrow [-1,1] \times (-1,1)/\!\sim\,,$$

where $(1,t) \sim (-1,-t)$. The latter space $\mathbb{M}^\circ = [-1,1] \times (-1,1)/\!\sim$ is the *open Moebius band*. Hence we have proved that if we take out a point (P) from the projective plane $\mathbb{RP}^2$ we obtain, up to homeomorphism, the open Moebius band $\mathbb{M}^\circ$. Consequently, since the projective plane is compact (because it is the image of a compact space) and since $\mathbb{RP}^2 - \{P\}$ is dense it, then $\mathbb{RP}^2$ is its Alexandroff compactification. We have proved the following result.

8.3.24 **Proposition**. *The Alexandroff compactification of the open Moebius band is the projective plane, i.e.* $(\mathbb{M}^\circ)^* \approx \mathbb{RP}^2$. □

8.3.25 Exercise. Using 8.3.23 or 8.3.24, show that the quotient of the (closed) Moebius band $\mathbb{M}$ obtained by collapsing the boundary $\partial\mathbb{M}$ onto a pointm is the projective plane. Here we define the boundary by $\partial\mathbb{M} = q([-1,1] \times \{-1,1\})$, if $q : [-1,1] \times [-1,1] \longrightarrow \mathbb{M}$ is the identification that defines $\mathbb{M}$, that is, $\mathbb{M}/\partial\mathbb{M} \approx \mathbb{RP}^2$.

8.4 Proper maps

In the previous section we defined the Alexandroff compactification of a non-compact topological space. In this section we shall study the behavior of this construction with respect to maps. We shall give several characterizations of the maps that can be extended to the Alexandroff compactifications. We shall call these maps proper. The reader can take a look at other books like [3], where alternative and complementary treatments of this topic are pursued. Here we start analyzing some examples. (In what follows, abusing notation, we denote by ∞ the added point in every space to obtain its compactification, but we shall understand that to different spaces different points ∞ are added.)

8.4.1 Examples.

(a) Take $X = (0,1)$, $Y = [0,1]$ and the inclusion $f : X \longrightarrow Y$. Clearly the Alexandroff compactification $X^* \approx \mathbb{S}^1$ with embedding $i : (0,1) \longrightarrow \mathbb{S}^1$ given by $i(t) = e^{2\pi i t}$. On the other hand, $Y^* = Y \sqcup \{\infty\}$. It is an easy exercise to observe that f does not admit an extension $f' : X^* \longrightarrow Y^*$.

(b) Moreover, if we consider $Z = \mathbb{R}$ and $g : X \longrightarrow Z$, then $g = j \circ f$, where j is the inclusion of $[0,1]$ en $\mathbb{R}$, then we have that both X and Z are not compact. In this case, an extension $g^* : X^* \longrightarrow Z^*$ of g does not exist either.

(c) Take now $X = Y = (0,1)$ and let $f : X \longrightarrow Y$ be given by $f(t) = \frac{1}{2}$ for all t. In this case, f does admit an extension $f' : X^* \longrightarrow Y^*$, namely the constant map with value $\frac{1}{2}$ or, the map $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ with value $-1 \in \mathbb{S}^1$, if we put $X^* = \mathbb{S}^1$. However, it is impossible to find an extension f^* such that $f^*(1) = 1$, that is, an extension that maps the added point ∞ to ∞.

The previous examples show that not every map between noncompact spaces can be extended canonically to their Alexandroff compactifications, mapping ∞ to ∞.

8.4.2 Exercise. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and let $f^* : \mathbb{R}^* \longrightarrow \mathbb{R}^*$ be given by $f^*(t) = f(t)$ if $t \in \mathbb{R}$ and $f^*(\infty) = \infty$. Show that f^* is continuous if and only if for every sequence $\{t_n\}$ in $\mathbb{R}$ such that $t_n \to \infty$, one has that $f(t_n) \to \infty$.

Let X and Y be topological spaces and take a continuous map $f : X \longrightarrow Y$. Define $f^* : X^* \longrightarrow Y^*$ by $f^*(x) = f(x)$ if $x \in X$ and $f^*(\infty) = \infty$. f^* Is continuous at $x \in X*$ if and only if for every neighborhood V of $f^*(x)$ in Y^*, the inverse image $(f^*)^{-1}(V)$ is a neighborhood of x in X^*. Clearly f^* is always continuous at $x \in X$ (*exercise*). When is f^* continuous at ∞?

Let V be an open neighborhood of ∞ in Y^*. Hence $Y^* - V$ is compact. We wish that $(f^*)^{-1}(V)$ is an open neighborhood of ∞ in X^*. That is, we want that $X^* - (f^*)^{-1}(V)$ is compact. This leads us to the following concept.

8.4.3 Definition. Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be a continuous map. We say that f is a *proper map* if for each compact set $K \subset Y$, the inverse image $f^{-1}(K) \subset X$ is a compact set too.

By the previous discussion, the following result must be clear.

8.4.4 **Theorem**. *Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be a continuous map. Then $f^* : X^* \longrightarrow Y^*$ is a continuous map if and only if f is a proper map.* □

Notice that the previous theorem is valid for any spaces X and Y, which can even be compact or not locally compact nor Hausdorff.

8.4.5 Note. Let X be a noncompact space and assume that $f : X \longrightarrow Y$ admits an extension $f' : X^* \longrightarrow Y$. In this case f cannot be proper since $f'(X^*) \subset Y$ is compact, but $X = f'^{-1}(f'(X^*))$ is not compact.

Let $f : X \longrightarrow Y$ be a proper map, i.e. $f^* : X^* \longrightarrow Y^*$ is continuous. If $A \subset X$ is closed, then $A \cup \{\infty\}$ is the complement of an open set in X^* and hence it is closed also in X^*. Thus $A \cup \{\infty\}$ is compact, so that

$$f^*(A \cup \{\infty\}) = f(A) \cup \{\infty\}$$

is also compact. If Y is a locally compact Hausdorff space, then Y^* is a compact Hausdorff space. Hence $f(A) \cup \{\infty\}$ is closed in Y^*. Since its complement is open, also the complement of $f(A)$ in Y is open. Therefore $f(A)$ is closed in Y. In particular, $f(X)$ is closed in Y and consequently $f(X)$ is locally compact. We have proved the next result.

8.4.6 **Proposition**. *Let Y be a locally compact Hausdorff space and let $f : X \longrightarrow Y$ be a proper map. Then f is a closed map and $f(X)$ is locally compact.* □

The latter proposition is false if one does not require, at least, that the compact sets of Y^* are closed.

8.4.7 Example. Let f be the inclusion of $X = \{x\}$ in $Y = \{x, y\}$, where Y has the indiscrete topology. Obviously f is proper. However, it is not a closed map.

The following result gives a useful characterization of proper maps.

8.4.8 **Theorem**. *Let Y be a Hausdorff locally compact space and let $f : X \longrightarrow Y$ be continuous. The following are equivalent:*

(a) *f is a proper map.*

(b) *f is a closed map and for every $y \in Y$, $f^{-1}(y)$ is compact.*

(c) *For every filter $\mathcal{F}$ in X and every cluster point y of the image filter $f(\mathcal{F})$, there exists a cluster point x of $\mathcal{F}$ such that $f(x) = y$.*

Proof:

(a)$\Longrightarrow$(b) by 8.4.6 and since every singular space $\{y\}$ is compact.

(b)$\Longrightarrow$(c) Let $\mathcal{F}$ be a filter in X and let $y \in Y$ be a cluster point of the image filter $f(\mathcal{F})$. Hence $y \in \overline{f(F)}$ for all $F \in \mathcal{F}$. Since f is a closed map, by 4.2.31(d), we have that $\overline{f(F)} = f(\overline{F})$. Thus $f^{-1}(y) \cap \overline{F} \neq \emptyset$ for every $F \in \mathcal{F}$. Therefore the set $\{f^{-1}(y) \cap \overline{F} \mid F \in \mathcal{F}\}$ is a filter basis in $f^{-1}(y)$ built up by closed sets. Since $f^{-1}(y)$ is compact, the filter basis has a cluster point $x \in f^{-1}(y)$, i.e. $x \in \overline{F}$ for every $F \in \mathcal{F}$. Consequently x is a cluster point of $\mathcal{F}$, as desired.

(c)$\Longrightarrow$(a) Let $K \subset Y$ be a compact set and let $\mathcal{G}$ be a filter in $f^{-1}(K)$ and $\mathcal{F}$ its extension to X. If we denote by $f_K(\mathcal{G})$ the image filter $f|_{f^{-1}(K)}$ of $\mathcal{G}$ in K, then the image of $\mathcal{F}$ under f in Y, $f(\mathcal{F})$, is its extension to Y. Since K is compact, $f|_{f^{-1}(K)}$ has a cluster point $y \in K$, which by 7.4.12(a) is also a cluster point of $f(\mathcal{F})$. By (c), there is a cluster point $x \in f^{-1}(y)$ of $\mathcal{F}$, which, again by 7.4.12(a), is also a cluster point of $\mathcal{G}$. Hence $f^{-1}(y)$ is compact and therefore f is proper. □

8.4.9 Note. The implication (a)$\Longrightarrow$(b) in the previous theorem is the only one that requires the assumption that Y is Hausdorff and locally compact. The implication (c)$\Longrightarrow$(b), analogously to what was done for (b)$\Longrightarrow$(c), can be shown without that assumption.

As a complement to Theorem 8.4.8 solve the next.

8.4.10 Exercise. Let Y be a Hausdorff locally compact space. Show that a map $f : X \longrightarrow Y$ is proper if and only if for any ultrafilter $\mathcal{U}$ in X and any limit point $y \in Y$ of the basis of image ultrafilter $f(\mathcal{U})$, there exists a limit point $x \in X$ of $\mathcal{U}$ such that $f(x) = y$. Since Y is Hausdorff, one has in fact that f is proper if and only if for any ultrafilter $\mathcal{U}$ in X such that the basis of the image ultrafilter $f(\mathcal{U})$ is convergent, one has that $\mathcal{U}$ itself is convergent.

There is another interesting characterization of a proper map.

8.4.11 **Theorem**. *Let Y be a Hausdorff locally compact space and let $f : X \longrightarrow Y$ be a continuous map. Then f is proper if and only if for any topological space Z the product map $f \times \mathrm{id}_Z : X \times Z \longrightarrow Y \times Z$ is a closed map.*

Proof: Assume first that f is proper. By 8.4.8(c), f has property 8.1.53(F). Therefore, by 8.1.54, $f \times \mathrm{id}_Z : X \times Z \longrightarrow Y \times Z$ has the same property, thus by 8.1.55, $f \times \mathrm{id}_Z$ is closed.

Conversely, let $K \subset Y$ be compact. If $f \times \mathrm{id}_Z : X \times Z \longrightarrow Y \times Z$ is a closed map, then the restriction

$$(f \times \mathrm{id}_Z)_{K\times Z} = f_K \times \mathrm{id}_Z : (f \times \mathrm{id}_Z)^{-1}(K \times Z) = f^{-1}(K) \times Z \longrightarrow K \times Z$$

is also a closed map (see 4.2.33). Since K is compact, by 8.1.56also the map $\mathrm{proj}_Z : K \times Z \longrightarrow Z$ is closed. Thus the composite

$$f^{-1}(K) \times Z \xrightarrow{f_K\times\mathrm{id}_Z} K \times Z \xrightarrow{\mathrm{proj}_Z} Z\,,$$

which coincides with $\mathrm{proj}_Z : f^{-1}(K) \times Z \longrightarrow Z$, is also a closed map. Again, by 8.1.56, we conclude that $f^{-1}(K)$ is compact. □

8.4.12 Exercise. Show that if X is a compact space, then any continuous map $f : X \longrightarrow Y$ is proper. Moreover, if P is a singular space, show that $g : X \longrightarrow P$ is proper if and only if X is compact.

8.4.13 Exercise. Let X and Y be topological spaces. Show that $\mathrm{proj}_Y : X \times Y \longrightarrow Y$ is proper if and only if X is compact.

8.4.14 Exercise. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps. Prove

(a) $\mathrm{id}_X : X \longrightarrow X$ is proper and, if f and g are proper, then also $g \circ f$ is proper.

(b) If $g \circ f$ is proper and f is surjective, then g is proper.

(c) If $g \circ f$ is proper and g is injective, then f is proper.

(d) If $g \circ f$ is proper and Y is Hausdorff, then f is proper.

8.4.15 Exercise. Assume that $f : X \longrightarrow Y$ is a continuous map, X is a compact space, and Y is a Hausdorff space. Show that f is proper.

8.4.16 Exercise. Show that the inclusion $(0, 1) \hookrightarrow \mathbb{R}$ is not proper. (Cf. 8.4.1(b).)

8.4.17 Exercise. Let $p_k : \mathbb{C} \longrightarrow \mathbb{C}$ be given by $p_k(z) = z^k$, $k \in \mathbb{Z}$. Is this map proper?

8.4.18 Exercise. Let X and Y be ininite discrete spaces. Under what conditions is a map $f : X \longrightarrow Y$ proper?

8.4.19 Exercise. A diagram of topological spaces and continuous maps

$$\begin{array}{ccc} G & \xrightarrow{F} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X \end{array}$$

is called *square!cartesian* or *pullback diagram* if

$$G = \{(y, e) \in Y \times E \mid f(y) = p(e)\}$$

(with the relative topology induced by the product topology) and $q(y, e) = y$, $F(y, e) = F$. Frequently G is denoted by $Y \times_X E$ and is called *fibered product* of Y and E over X. (See 5.3.6.) If X is a Hausdorff space, prove the following statements:

(a) If p is proper, then q is proper.

(b) If E is compact and f is proper, then G is compact.

(*Hint:* Notice that since X is Hausdorff, the diagonal $\Delta_X = \{(x, x) \mid x \in X\} \subset X \times X$ is closed.)

8.5 Compact-open topology

Let X and Y be topological spaces and take $\mathbf{Top}(X,Y) = \{f : X \longrightarrow Y \mid f \text{ is continuous}\}$. In this section we shall see how to furnish $\mathbf{Top}(X,Y)$ with a convenient topology. A natural function is

$$e : \mathbf{Top}(X,Y) \times X \longrightarrow Y\,,$$

given by $e(f,x) = f(x)$, which is called *evaluation*. If we provide $\mathbf{Top}(X,Y)$ with the discrete topology, then e is continuous. Which is the coarsest topology on $\mathbf{Top}(X,Y)$ such that the evaluation e is continuous? To face this question we formulate another one, which is more general.

Let X and Y be topological spaces and let Z be a set. Given a function of sets

$$\alpha : Z \times X \longrightarrow Y\,,$$

which topologies on Z make α continuous?

8.5.1 **Lemma**. *Assume that $\alpha : Z \times X \longrightarrow Y$ is continuous and let $\mathcal{N}_x$ be the neighborhood filter of x in X. Then*

(a) *The map $\alpha_z : X \longrightarrow Y$ given by $\alpha_z(x) = \alpha(z,x)$, is continuous for all z.*

(b) *If $\mathcal{F}$ is a filter in Z such that $\mathcal{F} \to z_0$, then $\alpha(\mathcal{F} \times \mathcal{N}_x) \to \alpha(z_0, x)$ for all $x \in X$, where $\mathcal{F} \times \mathcal{N}_x$ is the filter with the set of products of an element of $\mathcal{F}$ and an element of $\mathcal{N}_x$ as filter basis.*

Proof:

(a) This was already proved in Section 4.3.

(b) It is clear that $\mathcal{F} \times \mathcal{N}_x \to (z_0, x)$ (see 7.5.6). Since α is continuous we get the assertion. □

8.5.2 **Lemma**. *Take $H \subset \mathbf{Top}(X,Y)$ and $f_0 \in H$, and let $\mathcal{F}$ be a filter in H such that $e(\mathcal{F} \times \mathcal{N}_x) \to f_0(x)$ for all $x \in X$. Then there exists a topology on H for which $\mathcal{F}$ is the neighborhood filter of f_0 and therefore $\mathcal{F} \to f_0$, and for which e is continuous.*

Proof: Define a topology $\mathcal{A}$ on H by

$$\mathcal{A} = \{A \subset H \mid f_0 \notin A\} \cup \{A \in \mathcal{F} \mid f_0 \in A\}\,.$$

Clearly this topology is such that its neighborhood filter at f_0 in H, $\mathcal{N}^H_{f_0}$, is contained in $\mathcal{F}$, namely $\mathcal{F} \to f_0$.

We now see that e is continuous in topology. Take $f \in H$, $f \neq f_0$. For $x \in X$ take a neighborhood Q of $f(x)$ in Y. Since $f : X \longrightarrow Y$ is continuous, $f^{-1}(Q)$ is a neighborhood of x in X and $\{f\} \times f^{-1}(Q)$ is a neighborhood of (f, x) in $H \times X$. Moreover $e(\{f\} \times f^{-1}(Q)) = ff^{-1}(Q) \subset Q$. Therefore e is continuous at (f, x) if $f \neq f_0$.

The evaluation map e is also continuous at (f_0, x). Namely, by assumption $e(\mathcal{F} \times \mathcal{N}_x) \to f_0(x)$, we have that e maps the neighborhood filter of (f_0, x) to a convergent filter $e(f_0, x) = f_0(x)$. □

Take $H \subset \mathbf{Top}(X, Y)$, $f_0 \in H$, and a filter $\mathcal{F}$ in H. Assume that H has a topology for which $\mathcal{F} \to f_0$. Then $\mathcal{F} \to f_0$ for any coarser topology than the given one. Thus we have the following.

8.5.3 **Proposition**. *Assume there is a coarsest topology on H for which e is continuous. In this topology, $\mathcal{F} \to f_0$ if and only if $e(\mathcal{F} \times \mathcal{N}_x) \to f_0(x)$ for all $x \in X$.* □

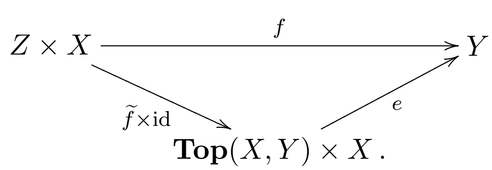
8.5.4 **Theorem**. *If $\mathbf{Top}(X, Y)$ has a coarsest topology for which the evaluation is continuous, then a function $f : Z \times X \longrightarrow Y$ is continuous if and only if the following conditions hold:*

(a) *For each point $z \in Z$, the function $f_z : X \longrightarrow Y$, given by $f_z(x) = f(z, x)$, is continuous.*

(b) *The function $\widetilde{f} : Z \longrightarrow \mathbf{Top}(X, Y)$, given by $\widetilde{f}(z) = f_z$, is continuous.*

Proof: We start showing that the conditions are sufficient.

By (a) we can decompose f as the following diagram shows.



By (b), $\widetilde{f} \times \mathrm{id}$ is continuous, and thus f is continuous too.

The two conditions are necessary. By Lemma 8.5.1(a), if f is continuous, so is f_z too. Therefore (a) holds.

To check (b), let $\mathcal{G}$ be a filter converging to z. Once more, Lemma 8.5.1(b) implies that for each point $x \in X$, $f(\mathcal{G} \times \mathcal{N}_z) \to f(z, x)$, namely $e(\widetilde{f} \times \mathrm{id})(\mathcal{G} \times \mathcal{N}_z) \to f(z, x) = e(\widetilde{f}(z), x)$. But $(\widetilde{f} \times \mathrm{id})(\mathcal{G} \times \mathcal{N}_z) = \widetilde{f}(\mathcal{G}) \times \mathcal{N}_x$, so that, by Proposition 8.5.3, $\widetilde{f}(\mathcal{G}) \to \widetilde{f}(z)$. Hence $\widetilde{f}$ is continuous. □

Take $H \subset \mathbf{Top}(X,Y)$ and assume that H has a topology such that the evaluation $e : H \times X \longrightarrow Y$ is continuous. Let $K \subset X$ be a compact set, let $Q \subset Y$ be an open set, and define $Q^K = \{f \in H \mid f(K) \subset Q\}$, namely $f \in Q^K$ if and only if $e(\{f\} \times K) \subset Q$.

8.5.5 **Proposition**. *$Q^K \subset H$ is an open set.*

Proof: Take $f \in Q^K$. For each point $x \in K$, Q is neighborhood de $f(x) = e(f,x)$. Since e is continuous, there exist neighborhoods $W(f,x)$ of f in H and $V(f,x)$ of x in X such that $e(W(f,x) \times V(f,x)) \subset Q$. Since K is compact, one can cover it with finitely many of the neighborhoods $V(f,x)$. Say $V(f,x_1), \dots V(f,x_n)$ cover K. Thus $W = W(f,x_1) \cap \cdots \cap W(f,x_n)$ is a neighborhood of f in H and $e(W \times V(f,x_i)) \subset Q$, for $i = 1, \dots, n$. Since $K \subset V(f,x_1) \cup \cdots \cup V(f,x_n)$, one has $e(W \times K) \subset Q$, i.e. $W \subset Q^K$. This shows that Q^K is open. □

We have shown that if a topology on H is such that the evaluation map e is continuous, then the sets Q^K must be open in this topology. Thus even the coarsest topology on H such that e is continuous must contain these sets. This suggests the following.

8.5.6 Definition. Take $H \subset \mathbf{Top}(X,Y)$. The topology in H that has the family

$$\{Q^K \mid K \subset X \text{ is compact and } Q \subset Y \text{ is open}\}$$

as a subbasis, is called the *compact-open* topology in H. We shall use the notation $\mathbf{M}(X,Y)$ for $H = \mathbf{Top}(X,Y)$ with this topology.

8.5.7 Note. The sets Q^K do not form in general a basis for the topology of $\mathbf{M}(X,Y)$.

By construction of the compact-open topology we have the following.

8.5.8 **Lemma**. *The compact-open topology in $H \subset \mathbf{M}(X,Y)$ is coarser than any topology for which the evaluation $e : H \times X \longrightarrow Y$ is continuous.* □

Let X be locally compact. Take $(f_0, x_0) \in H \times X$, and an open neighborhood Q of $f_0(x_0)$ in Y. Then by the continuity of f_0, there exists a compact neighborhood K of x in X such that $f_0(K) \subset Q$. Hence $f_0 \in Q^K$, and thus Q^K is a neighborhood of f_0 in the compact-open topology. Hence $Q^K \times K$ is a neighborhood of (f_0, x_0) and by definition, $e(Q^K \times K) \subset Q$. Therefore $e : H \times X \longrightarrow Y$ is continuous with respect to the compact-open topology en H. By the previous lemma, we have shown the next.

8.5.9 **Theorem**. *Let X be a locally compact space and let Y be any space. Then for any $H \subset \mathbf{M}(X,Y)$, the compact-open topology in H is the coarsest for which the evaluation $e : H \times X \longrightarrow Y$ is continuous.* □

8.5.10 **Theorem**. *Let X be a locally compact space and let Y be any space. Take $Y' = \{f \in \mathbf{M}(X,Y) \mid f$ is constant$\}$, then there is a homeomorphism $Y \approx Y'$. Moreover, $\mathbf{M}(X,Y)$ is a Hausdorff space if and only if Y is Hausdorff too.*

Proof: The map $i : Y \longrightarrow \mathbf{M}(X,Y)$ given by $i(y) : x \mapsto y$ for all $x \in X$, is an embedding whose image is Y'. Its inverse is given by $f \mapsto f(x) = e(f,x)$ for some $x \in X$. Therefore, if $\mathbf{M}(X,Y)$ is a Hausdorff space, then Y' is a Hausdorff space and thereby Y is also a Hausdorff space.

Conversely, if Y is a Hausdorff space, take $f, g \in \mathbf{M}(X,Y)$, $f \neq g$. Therefore, for some $x \in X$, one has $f(x) \neq g(x)$. Thus there are open neighborhoods Q_1 and Q_2 of $f(x)$ and $g(x)$ in Y, respectively such that $Q_1 \cap Q_2 = \emptyset$. Hence the sets $Q_1^{\{x\}}$ and $Q_2^{\{x\}}$ are open and disjoint, since if $h \in Q_1^{\{x\}} \cap Q_2^{\{x\}}$, $h(x) \in Q_1 \cap Q_2$, which is impossible. Clearly $f \in Q_1^{\{x\}}$ and $g \in Q_2^{\{x\}}$. Hence $\mathbf{M}(X,Y)$ is a Hausdorff space. □

To finish this section, we shall give a very interesting application of the compact-open topology, that occurs when the domain space is locally compact. First we notice in the next example, due to Dieudonné, that in general the product of identifications is not necessarily an identification.

8.5.11 Example. Let $\mathbb{Q}$ denote the set of rational numbers with the relative topology and take relation $\sim$ on $\mathbb{Q}$ that identifies all integers in one point. Let $p : \mathbb{Q} \longrightarrow \mathbb{Q}/\!\sim$ be the quotient map, which is an identification. However the product map

$$p \times \mathrm{id} : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}/\!\sim \times \mathbb{Q}$$

is not an identification.

8.5.12 Exercise. Prove that in the previous example, the identification p is not an open map, but it is indeed a closed map. Moreover prove that in fact the product map $p \times \mathrm{id}_{\mathbb{Q}}$ is not an identification.

The next result, due to J. H. C. Whitehead, is a strong application of compactness. It solves the problem of when the product of identifications is again an identification. As we saw in Example 8.5.11, this in general not true.

8.5.13 **Theorem**. *Let $p : X \longrightarrow X'$ be an identification and let Y be a locally compact space. Then the product map*

$$p \times \mathrm{id} : X \times Y \longrightarrow X' \times Y$$

is an identification.

Proof: Take an equivalence relation in $X \times Y$ given by $(x, y) \sim (x', y)$ if and only if $p(x) = p(x')$ and let $q : X \times Y \longrightarrow Z = X \times Y/\!\sim$ be the quotient map.

By the universal property of the quotient maps and since $p \times \mathrm{id}$ and q identify the same points, there exists a unique continuous bijective map $h : Z = X \times Y/\!\sim \longrightarrow X' \times Y$. By 8.5.4(b) and 8.5.9, q determines a map $\widetilde{q} : X \longrightarrow \mathbf{M}(Y, Z)$, continuous with respect to the compact-open topology on $\mathbf{M}(Y, Z)$. Moreover, if $p(x) = p(x')$, then clearly $\widetilde{q}(x) = \widetilde{q}(x')$. Thus, since p is an identification, one has a continuous map $q' : X' \longrightarrow \mathbf{M}(Y, Z)$ such that

$$q' \circ p = \widetilde{q}\,.$$

Since the evaluation $e : \mathbf{M}(Y, Z) \times Y \longrightarrow Z$ is continuous and Y is locally compact, the function $e \circ q' : X' \times Y \longrightarrow Z$, which coincides with h^{-1}, is continuous. Hence h is a homeomorphism and thus $p \times \mathrm{id}$ is also an identification. □

In Section 8.7 (8.7.21) below we shall state a generalization of this result.

8.6 The exponential law

Given arbitrary sets X and Y, we denote (provisionally) by X^Y the set of functions $f : X \longrightarrow Y$. If X, Y, Z are sets, then the *exponential law* establishes an equivalence of sets

$$Z^{X \times Y} \cong (Z^Y)^X\,.$$

For these, we only have to define $\varphi : Z^{X \times Y} \to (Z^Y)^X$ by $\varphi(f)(x)(y) = f(x, y)$ and, as an inverse, $\psi : (Z^Y)^X \to Z^{X \times Y}$ by $\psi(g)(x, y) = g(x)(y)$.

Now we try to establish an analogous result for the space $\mathbf{M}(X, Y)$, if X and Y are topological spaces.

8.6.1 **Proposition**. *Let X, Y, Z topological spaces such that Y is Hausdorff and locally compacto. Then there is a set-equivalence*

$$\varphi : \mathbf{M}(X \times Y, Z) \to \mathbf{M}(X, \mathbf{M}(Y, Z))\,.$$

Proof: To define φ, as we did at the beginning of the section, we have to prove that if $f : X \times Y \to Z$ is continuous, then $\varphi(f)(x) : Y \to Z$ is continuous and that $\varphi(f) : X \to \mathbf{M}(Y, Z)$ is also continuous.

For the first statement, observe that $\varphi(f)(x)$ is the composite

$$Y \xrightarrow{i_x} X \times Y \xrightarrow{f} Z\,,$$

where $i_x(y) = (x, y)$, which is clearly continuous. (Notice that if $X = \emptyset$, then the result is trivial.)

For the second statement, let U^K be a subbasic open set of $\mathbf{M}(Y, Z)$. It is enough to see that $\varphi(f)^{-1}(U^K)$ is open in X. Thus take $x \in \varphi(f)^{-1}(U^K)$. Then $\varphi(f)(x)(y) = f(x, y) \in U$ for all $y \in K$ and there are neighborhoods W_y of x and V_y of y such that $f(W_y \times V_y) \subset U$. Since K is compact, the family $\{V_y\}$ contains a finite subfamily $V_1, \dots, V_m$ which covers K. Take $W = W_1 \cup \cdots \cup W_m$, where W_i is such that $f(W_i \times V_i) \subset U$. W is a neighborhood of x in X. Let us see now that $W \subset \varphi(f)^{-1}(U^K)$. Indeed, if we take $x' \in W$ and $y \in K$, then $\varphi(f)(x')(y) = f(x', y)$, but one has $y \in V_i$ for some i, and $x' \in W_i$, hence $f(x', y) \in U$.

We have thus shown that φ is well defined.

Let us see now that with the definition given at the beginning of the section, the map

$$\psi : \mathbf{M}(X, \mathbf{M}(Y, Z)) \to \mathbf{M}(X \times Y, Z)$$

is well defined. Take $g : X \to \mathbf{M}(Y, Z)$ continuous. It is enough to show that $\psi(g)$ is continuous.

To see this, take an open set $U \subset Z$ and we show that $\psi(g)^{-1}(U)$ is open. Take $(x, y) \in \psi(g)^{-1}(U)$, i.e. $g(x)(y) \in U$. Since $g(x)$ is continuous, there is a neighborhood W of y such that $g(x)(W) \subset U$. Since Y locally compact and Hausdorff, there is a compact neighborhood K such that $y \in K \subset W$ Hence $g(x)(K) \subset U$ and therefore $g(x) \in U^K$), which is open in $\mathbf{M}(Y, Z)$.

Since g is continuous, there is a neighborhood V of x in X such that $g(V) \subset U^K$. $V \times K$ is a neighborhood of (x, y) en $X \times Y$. Take $(x', y') \in V \times K$. Then $\psi(g)(x', y') = g(x')(y') \in U$ and thus $V \times K \subset \psi(g)^{-1}(U)$. □

With an additional condition the set-equivalence in the previous result is a homeomorphism. I.e. we obtain the *exponential law*.

8.6.2 **Theorem**. *If X, Y, Z ar topological spaces such that X and Y are Hausdorff spaces and Y is locally compact, then*

$$\varphi : \mathbf{M}(X \times Y, Z) \to \mathbf{M}(X, M(Y, Z))$$

is a homeomorphism.

Proof: We see that φ and ψ are continuous.

First take a subbasic set $(U^L)^K$ in $\mathbf{M}(X, \mathbf{M}(Y, Z))$ with U open in Z and K and L compact in X and Y, respectively. Then $K \times L$ is compact, and if $f \in (U^{K\times L}) \subset \mathbf{M}(X \times Y, Z)$, then $\varphi(f)(K)(L) = f(K \times L) \in U$- I.e. $\varphi(U^{K\times L}) \subset (U^L)^K$.

Let now U^J be a subbasic set in $\mathbf{M}(X \times Y, Z)$, with J compact in $X \times Y$. Take $K = \mathrm{proj}_X(J)$ and $L = \mathrm{proj}_Y(J)$. K and L are compact and $J \subset K \times L$. Let us see $\psi((U^L)^K) \subset U^J$. Indeed, take $g \in (U^L)^K$ and $(x, y) \in J$, then $\psi(g)(x, y) = g(x)(y) \in U$, since $x \in K$ and $y \in L$. □

We have the map

$$\mathbf{M}(X, Y) \times \mathbf{M}(Y, Z) \to \mathbf{M}(X, Z) \tag{8.6.3}$$

given by composition.

8.6.4 EXERCISE. Prove that if X and Y are Hausdorff locally compact spaces, then the map (8.6.3) is continuous. In particular, if $f : X \to Y$ is continuous, then (by restriction of (8.6.3)) it induces a continuous map

$$f^\# : \mathbf{M}(Y, Z) \to \mathbf{M}(X, Z)\,,$$

given by $f^\#(g) = g \circ f$. Similarly, if $g : Y \to Z$ is continuous, then it induces (again by restriction of (8.6.3)) a continuous map

$$g_\# : \mathbf{M}(X, Y) \to \mathbf{M}(X, Z)$$

such that $g_\#(f) = g \circ f$.

8.6.5 DEFINITION. Let A be a subspace of X and B a subspace of Y. Denote by $\mathbf{M}(X, A; Y, B)$ the subspace of $\mathbf{M}(X, Y)$ consisting of the maps $f : X \to Y$ such that $f(A) \subset B$. An important instance of these subspaces is the space $\mathbf{M}(X, x_0; Y, y_0)$ of maps $f : X \to Y$ such that $f(x_0) = y_0$, if $x_0 \in X$ and $y_0 \in Y$ are specified points. These maps are called *pointed maps* (or (o *based maps*), since they map the *base point* x_0 of X to the base point y_0 of Y. The pairs (X, x_0) or (Y, y_0) are called *pointed spaces.*

8.6.6 EXAMPLE. Let $I = [0, 1]$ be the interval and let $\partial I = \{0, 1\}$ be its boundary. Thus we can consider the spaces

$$\mathbf{M}(I, X) \supset \mathbf{M}(I, 0; X, x_0) \supset \mathbf{M}(I, \partial I; X, x_0)\,,$$

for a pointed space (X, x_0). These spaces are known as *free path-space in* X, *path-space in* X *based in* x_0, and *loop-space of* X *based at* x_0, respectively. The set $\mathbf{M}(I, \partial I; X, x_0)$ is usually denoted by $\Omega(X, x_0)$ or if the base point is clear, then by ΩX (compare with 8.6.9).

8.6.7 DEFINITION. Let us consider two *pairs of spaces* (X, A) and (Y, B). We define its *product* as the pair of spaces

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)\,.$$

Hence $(I, \partial I) \times (I, \partial I) = (I^2, \partial I^2)$, where I^2 is the unit space in the plane and ∂I^2 its boundary, which is homeomorphic to the circle $\mathbb{S}^1$ (see Figure 8.2).

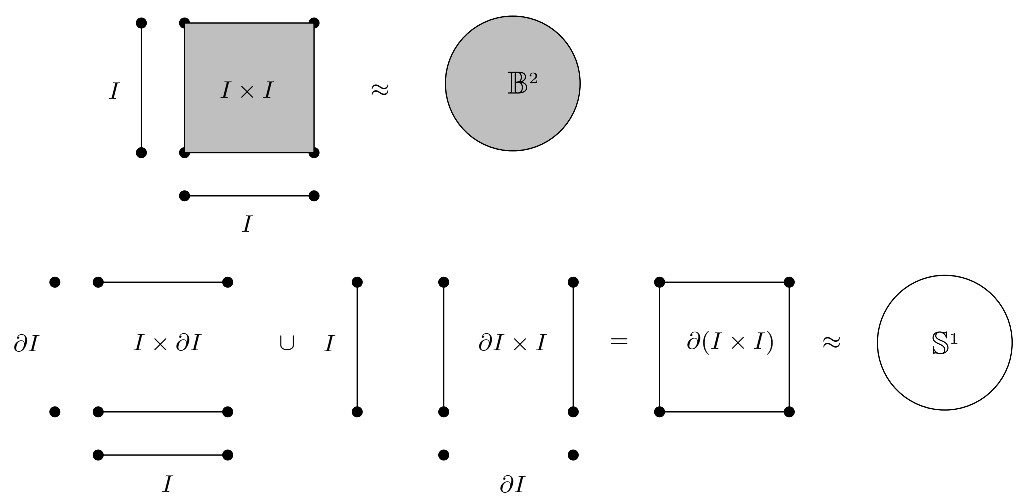


Figure 8.2 $(I, \partial I) \times (I, \partial I) = (I^2, \partial I^2) \approx (\mathbb{B}^2, \mathbb{S}^1)$

Inductively, $(I^n, \partial I^n) \times (I, \partial I) = (I^{n+1}, \partial I^{n+1})$, where I^{n+1} is the unit cube in $\mathbb{R}^{n+1}$ and ∂I^{n+1} is its boundary, que is homeomorphic to the sphere

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

By the exponential law (which is also valid for pairs −*exercise*), we have

$$\mathbf{M}(I^{n+1}, \partial I^{n+1}; X, x_0) \approx \mathbf{M}(I, \partial I; \mathbf{M}(I^n, \partial I^n; X, x_0), \widetilde{x_0})\,, \tag{8.6.8}$$

where $\widetilde{x_0} \in \mathbf{M}(I^n, \partial I^n; X, x_0)$ is such that $\widetilde{x_0}(I^n) = x_0$.

8.6.9 DEFINITION. The space $\mathbf{M}(I^n, \partial I^n; X, x_0)$ is called the *n-loop space in* X *based in* x_0 and and is denoted by

$$\Omega^n(X, x_0)\,.$$

If the base point is clear, then we abuse the notation and write $\Omega^n X$.

By (8.6.8) we have

$$\Omega(\Omega^n(X, x_0), \widetilde{x_0}) \approx \Omega^{n+1}(X, x_0).$$

8.6.10 Exercise. Let X be a pointed space. Show that there is a homeomorphism

$$\Omega^n(X, x_0) \approx \mathbf{M}(\mathbb{S}^n, *; Z, x_0).$$

8.7 Compactly generated spaces

For several constructions of new topological spaces out of old ones, it it is convenient to have an adequate topology on the spaces, in order to have better properties. In this section we shall study a certain class of spaces, whose topology is determined by their compact subsets. We shall also how to modify a given topology on any Hausdorff space so that it becomes a space with this new topology. This topology does not alter very much the given topology, so that using it we may deduce many properties of the original one, particularly homotopy theoretical properties of the space. This topology was introduced by Kelley [12] and was studied in detail by Steenrod [18], whose ideas we partly follow. Other references for properties of this class of spaces are [4] and [5]. There are other convenient classes of topological spaces which are similar to but more general than this one. At the end of the section we shall say some words about them.

8.7.1 Definition. A Hausdorff space X is said to be *compactly generated* if the following axiom holds:

(CG) A subset $A \subset X$ is closed if and only if $A \cap K$ is closed for any compact subset $K \subset X$.

In other words, a space is compactly generated if and only if it has the *weak topology* generated by its compact subspaces, i.e. the finest topology for which the inclusions $K \hookrightarrow X$ are continuous for all compact subsets $K \subset X$. I.e. we have the following universal property.

8.7.2 **Theorem**. *Let X be a Hausdorff space. Then X is compactly generated if and only if it satisfies the following condition:*

For every space Y and any map $f : X \longrightarrow Y$, f is continuous if and only if $f|_K$ is continuous for each compact subset $K \subset X$.

Proof: Assume X is compactly generated and take a map $f : X \longrightarrow Y$. If f is continuous, then $f|_K$ Is continuous too for every compact set K.

Conversely, if $f|_K$ is continuous para every compact subset $K \subset X$ and $D \subset Y$ is a closed set, then $(f^{-1}D) \cap K = (f|_K)^{-1}(D)$ is closed in K. Hence, since X is compactly generated, $f^{-1}D$ is closed and thus f is continuous.

Assume now that X is a Hausdorff space and satisfies the condition, and take $A \subset X$ such that for each compact set $K \subset X$, $A \cap K$ is closed. Let $\widetilde{X}$ be the space with the same underlying set as X, but its closed sets are precisely those sets B such that $B \cap K$ is closed in X for each compact subset $K \subset X$. Hence A is closed in $\widetilde{X}$. On the other hand, by the very definition of $\widetilde{X}$, the identity map $\mathrm{id} : X \longrightarrow \widetilde{X}$ is such that $\mathrm{id}|_K$ is continuous. Hence by the condition id is continuous and therefore $A = \mathrm{id}^{-1}A$ is closed in X. Thus X is compactly generated. □

8.7.3 EXERCISE. Prove that $\widetilde{X}$, as defined in the previous proof, is indeed a topological space, namely, that the set of its closed sets satisfies the axioms for closed sets 2.2.5. Moreover prove that $\widetilde{X}$ is a compactly generated space. (*Hint:* Check that $\widetilde{X}$ and X have the same compact sets.)

The next result gives a criterion, i.e. a sufficient condition for a space to be compactly generated.

8.7.4 **Theorem**. *Let X be a Hausdorff space such that for every subset $A \subset X$ and for each point $x \in \overline{A}-A$, there is a compact set $K \subset X$ such that $x \in \overline{A \cap K}-A \cap K$. Then X is compactly generated.*

Proof: Let $A \subset X$ be such that for every compact set $K \subset X$, the intersection $A \cap K$ is closed, and let x be a cluster point of A. By assumption there is a compact set $K_0 \subset X$ such that x is a cluster point of $A \cap K_0$. Since this set is closed, $x \in A \cap K_0 \subset A$. Thus A is closed. Hence X is compactly generated. □

The converse of the previous result is false, as the following example shows.

8.7.5 EXAMPLE. Let Y be the space whose points are all ordinals less than or equal to Ω, the first uncountable ordinal. Let Y have the order topology, i.e. the topology which has the half lines $I_a = \{y \in Y \mid y < a\}$ and $I^b = \{y \in Y \mid b < y\}$, $a, b \in Y$, as subbasis. Now take as X the subspace of Y obtained by deleting all limit ordinals (i.e. those ordinals without an immediate predecessor), except Ω. Thus $K \subset X$ is compact if and only if K is finite, since every infinite subset of X contains a sequence that converges to a limit ordinal different from Ω. Consequently $A = X - \{\Omega\}$ meets each compact set in a closed set, However A is not closed, since $\Omega \in \overline{A}$. Therefore X is not compactly generated.

On the other hand Y is compactly generated. It is indeed compact. This shows that not every (open) subspace of a compactly generated space is compactly generated.

Before we analyze when the property of a space being compactly generated is inherited by subspaces, we shall see that the class of compactly generated spaces is ample, so that many of the spaces that come from applications of topology are of this sort. We shall do this using criterion 8.7.4.

8.7.6 **Proposition**. *Every locally compact Hausdorff space X is compactly generated.*

Proof: Let $x \in X$ be a cluster point of a set $A \subset X$ and teke a compact neighborhood K of x. Then the intersection $A \cap K \neq \emptyset$ and x is a cluster point of it. By 8.7.4, X is compactly generated. □

8.7.7 **Theorem**. *Let X be a Hausdorff space. Then X is compactly generated if and only if X is a quotient of a locally compact space.*

Proof: Assume that X is compactly generated. Since it has the weak topology induced by its compact subspaces, it is a quotient of the topological sum of all of them. Namely,

$$q : Y = \coprod_{K \subset X\,,\; K \text{ compact}} K \longrightarrow X\,,$$

given by the inclusions $q|K = i_K : K \hookrightarrow X$, is an identification. Obviously Y is locally compact.

Conversely, if there is an identification $q : Y \longrightarrow X$ with Y locally compact, take $A \subset X$ such that $A \cap C$ is open in K for each compact set $K \subset X$. We shall see that A is open en X. If $V \subset Y$ is an open set such that $\overline{V}$ is compact, then $A \cap q(\overline{V}) = q(\overline{V}) \cap G$ for some open set G in X. Since $q^{-1}(A) \cap q^{-1}q(\overline{V}) = q^{-1}q(\overline{V}) \cap q^{-1}(G)$, if we intersect it with V, then we have that $q^{-1}(A) \cap V = V \cap q^{-1}(G)$. Hence $q^{-1}(A) \cap V$ is open in Y. Since Y is locally compact, we can cover it with a family $\{V_\alpha\}$ of relatively compact open sets, in such a way that $q^{-1}(A) = \bigcup_\alpha (q^{-1}(A) \cap V_\alpha)$. This shows that $q^{-1}(A)$ is open in Y and since q is an identification, A is open en X. □

Similarly to 8.7.6 we have the following.

8.7.8 **Proposition**. *Every first-countable Hausdorff space is compactly generated.*

Proof: Let X be first-countable and let x be a cluster point of a set $A \subset X$. Hence there is sequence $x_n \in A$ such that $x_n \to x$. Put $K = \{x_n\} \cup \{x\}$. Hence the intersection $A \cap K \neq \emptyset$ and x is a cluster point of it. By 8.7.4, X is compactly generated. □

We shall see now that the property of being compactly generated is inherited by the closed subspaces and by some open subspaces.

8.7.9 Definition. Let X be a topological space and take $Y \subset X$. We say that Y is a *regular open set* if every point $y \in Y$ has a neighborhood V in X such that $\overline{V} \subset Y$.

8.7.10 **Proposition**. *Let X be a compactly generated space and take $Y \subset X$. If Y is closed or regular open, then Y, furnished with the relative topology, is compactly generated.*

Proof: Assume that Y is closed and take $B \subset Y$ such that $B \cap K$ is closed for every compact set $K \subset Y$. If $K' \subset X$ is compact, then $B \cap K' = B \cap (K' \cap Y)$ is closed since $(K' \cap Y) \subset Y$ is compact. Therefore B is closed in X and thus it is closed in Y too.

Assume now that Y is regular open and take $B \subset Y$ such that $B \cap K$ is closed in Y for every compact set $K \subset Y$. Let y be a cluster point of B in Y. Hence there is a neighborhood V of y in X such that $\overline{V} \subset Y$. If $K' \subset X$ is compact, then $\overline{V} \cap K' \subset Y$ is compact. Consequently $B \cap \overline{V} \cap K'$ is closed in Y, is closed in $\overline{V} \cap K'$ and hence also in X. Therefore, since K' is arbitrary and X is compactly generated, we have that $B \cap \overline{V}$ is closed in X. Finally, since y is a cluster point of B, it is also a cluster point of $B \cap \overline{V}$. Hence, since $B \cap \overline{V}$ is closed, $y \in B \cap \overline{V} \subset B$, i.e. $y \in B$. Hence B is closed. □

8.7.11 Exercise. Show that the definition of regular open set given above coincides with the one given in Exercise 2.2.22.

The assumption that Y is regular open in X in the previous proposition cannot be replaced by the assumption that Y is open, as we showed in Example 8.7.5.

Dually we can ask if an identification space of a compactly generated space is a compactly generated space. First we notice that there is a necessary condition for it is the fact that the resulting space has to be Hausdorff, which is not always the case. Indeed this condition is also sufficient.

8.7.12 **Proposition**. *Let X be a compactly generated space and let $q : X \longrightarrow Z$ be an identification such that Z is Hausdorff. Then Z is compactly generated.*

Proof: Since X compactly generated, by 8.7.7 there is a locally compact space Y and an identification $p : Y \longrightarrow X$. Hence the composite $q \circ p : Y \longrightarrow Z$ is identification too, and again by 8.7.7, Z is compactly generated. □

Given any Hausdorff space X, as we saw in the proof of 8.7.2, there is a space associated to X, which we called $\widetilde{X}$, which is compactly generated.

8.7.13 Definition. Let X be a Hausdorff space. Define $c(X)$ as the space with the same underlying space as X but furnished with the topology whose family of closed sets is

$$\mathcal{C} = \{A \subset X \mid A \cap K \text{ is closed in } X \text{ for all compact subsets } K \subset X\}\,.$$

As it was asked in Exercise 8.7.3, this family $\mathcal{C}$ satisfies the axioms for the closed sets of topological space, and thus it makes $c(X)$ a compactly generated space which has exactly the same compact sets of X. We call $c(X)$ the *compactly generated space associated to X* .

8.7.14 Remark. Usually what we denote by $c(X)$ is denoted by $k(X)$. However we avoid this notation, since $k(X)$ will denote a similar but more general construction, that we present below in 8.8.1.

This construction is *functorial*, namely, given any map $f : X \longrightarrow Y$, the same function between sets determines a continuous map

$$c(f) : c(X) \longrightarrow c(Y)$$

with the following properties:

(a) $c(\mathrm{id}_X) = \mathrm{id}_{c(X)} : c(X) \longrightarrow c(X)$, if X is a Hausdorff space, and

(b) $c(g \circ f) = c(g) \circ c(f) : c(X) \longrightarrow c(Z)$, if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous maps between Hausdorff spaces.

8.7.15 Exercise. Show that indeed $c(f) : c(X) \longrightarrow c(Y)$ is continuous.

The following result summarizes the fundamental properties of $c(X)$.

8.7.16 **Theorem**. *Let X and Y be Hausdorff spaces. Then $c(X)$ has the following properties.*

(a) *The identity map $j : c(X) \longrightarrow X$ is continuous.*

(b) *$c(X)$ is a Hausdorff space.*

(c) *X and $c(X)$ have the same compact sets.*

(d) *$c(X)$ is compactly generated.*

(e) *If X is compactly generated, then $c(X) = X$.*

(f) *If $f : X \longrightarrow Y$ is continuous on each compact subspace of X, then $c(f) : c(X) \longrightarrow c(Y)$ is continuous.*

Proof:

(a) It is clear since any closed set in X is also closed in $c(X)$.

(b) It is clear too since the neighborhoods in X are neighborhoods also in $c(X)$ (the topology of $c(X)$ is finer than that of X).

(c) If $K \subset c(X)$ is compact, then by (a) $K \subset X$ is compact too (it is the image under j of a compact set). Conversely, let $K \subset X$ be compact and let $\widetilde{K} \subset c(X)$ be the same set K with the relative topology. By (a), the identity $\widetilde{K} \longrightarrow K$ is continuous. To see that its inverse map is also continuous, let $B \subset K$ be closed. Obviously, B meets each compact subset of X in a closed set. Therefore B is closed in $c(X)$ and hence also in $\widetilde{K}$. This proves that the identity $X \longrightarrow \widetilde{X}$ is continuous. Hence $\widetilde{X}$ is compact.

(d) If A meets each compact subset of $c(X)$ in a closed set, then by (c) A meets any compact subset of X in a compact set, thus also in a closed set. Thus A is closed in $c(X)$.

(e) It follows directly from (d).

(f) Let $B \subset c(Y)$ be a closed set. Then $B \cap L$ is closed in Y for each compact set L. Let $K \subset X$ be compact. Then $f^{-1}(B) \cap K = (f|_K)^{-1}(B \cap f(K))$ is closed in K and hence also in X, since $f(K) \subset Y$ is compact. This proves that $f^{-1}(B)$ is closed in $c(X)$, and thus $c(f) : c(X) \longrightarrow c(Y)$ is continuous. $\square$

There is a universal property that que characterizes the construction $c(X)$.

8.7.17 **Proposition**. *Let X be a Hausdorff space. The identity map $c(X) \longrightarrow X$ has the following universal property that characterizes it.*

(CG) *Let Y be a compactly generated space and $f : Y \longrightarrow X$ a continuous map, then there is a unique continuous map $\widehat{f} : Y \longrightarrow c(X)$ such that the following diagram commutes:*

$$\begin{array}{ccc} c(X) & \longrightarrow & X \\ {\scriptstyle \widehat{f}}\uparrow & \nearrow_{f} & \\ Y & & \end{array} .$$

This unique map $\widehat{f}$, as a function of sets is f.

Proof: Let us see that $c(X) \longrightarrow X$ satisfies (CG). For this, take $\widehat{f}$ equal to f as a function. Since Y is compactly generated, by 8.7.16(e), $c(Y) = Y$, so that $\widehat{f} = c(f)$ is continuous. It is clear that $\widehat{f}$ is unique and that the diagram commutes.

Conversely, if $\widetilde{X}$ is compactly generated and the map $i : \widetilde{X} \to X$ satisfies (CG), then for the identity $j : c(X) \longrightarrow X$ there is a unique map $\widehat{j} : c(X) \longrightarrow \widetilde{X}$ such that $i \circ \widehat{j} = j$. Analogously, since $j : c(X) \longrightarrow X$ satisfies (CG), there is a unique map $\widehat{i} : \widetilde{X} \longrightarrow c(X)$ such that $j \circ \widehat{i} = i$. Applying twice the uniqueness required by (CG), we conclude that $\widehat{j} \circ \widehat{i} = \mathrm{id}_{\widetilde{X}}$ and that $\widehat{i} \circ \widehat{j} = \mathrm{id}_{c(X)}$. I.e. $\widetilde{X} \approx c(X)$. □

Let $p : X \longrightarrow X'$ and $q : Y \longrightarrow Y'$ be identifications. As shown in Example 8.5.11, the product

$$p \times q : X \times Y \longrightarrow X' \times Y'$$

is not always an identification. In the previous section (8.5.13) we proved a result that under some assumption it guarantees that the product of two identifications is again an identification. This assumption is the local compactness of one of the factors. One of the advantages of working with the class of compactly generated spaces is the fact that, redefining the product topology adequately, the product of identifications remains an identification without further assumptions on the spaces in question. This way, the class of spaces where this result is valid is much larger than the original class where we already had the result.

On the other hand, an example of Dowker's [6] shows that the product of two compactly generated spaces need not be compactly generated (in fact, what Dowker shows is that the product of two *CW-complexes*, see [1] or [15], which by definition are compactly generated, is not necessarily a CW-complex. This happens precisely because the product is not compactly generated). For that reason, it is convenient to modify the definition of topological product.

8.7.18 Definition. Let X and Y be compactly generated spaces. Their *compactly generated product* or their *k-product* is defined by

$$X \widetilde{\times} Y = c(X \times Y)\,.$$

Indeed, this is a good definition, since it has the universal property of the product in the class of compactly generated spaces, as the next result shows.

8.7.19 **Theorem**. *The projections* $p : X\widetilde{\times}Y \longrightarrow X$ *and* $q : X\widetilde{\times}Y \longrightarrow Y$ *are continuous and if* Z *is a compactly generated space and we have a map* $f : Z \longrightarrow X\widetilde{\times}Y$*, then* f *is continuous if and only if the maps* $p \circ f$ *and* $q \circ f$ *are continuous.*

Proof: Since the projections of the topological product $X \times Y$ are continuous, then by 8.7.16(a), p and q are continuous.

Let Z be a compactly generated space and take $f : Z \longrightarrow X\widetilde{\times}X$. If f is continuous, then also the composites $p \circ f$ and $q \circ f$ are continuous. Conversely, if these composites are continuous, then by the universal property of the usual topological product, the map $f : Z \longrightarrow X \times Y$ is continuous. Hence, since Z compactly generated, by 8.7.17, f, as a map $Z \longrightarrow X\widetilde{\times}Y$, is continuous. □

8.7.20 **Proposition**. *If* X *is a locally compact space and* Y *is a compactly generated space, then* $X\widetilde{\times}Y = X \times Y$*.*

Proof: Let $A \subset X \times Y$ be such that for every compact set $K \subset X \times Y$, $A \cap K$ is closed and take $(x, y) \in X \times Y - A$. Since X is locally compact, x has a neighborhood V such that its closure $\overline{V}$ is compact. Since $\overline{V} \times \{y\}$ is also compact, $A \cap \overline{V} \times \{y\}$ must be closed. Thus x has a neighborhood U smaller than V such that $\overline{U} \times \{y\}$ does not meet A.

Let now B be the image in Y of $A \cap (\overline{U} \times Y)$ under the projection. If $C \subset Y$ is compact, then $A \cap (\overline{U} \times C)$ is compact. Hence $B \cap C$ is closed. Since Y is compactly generated, B must be closed in Y. But y does not lie in B, therefore $U \times (Y - B)$ is una neighborhood de (x, y) which does not meet A. Consequently, A is closed in $X \times Y$, and so $X \times Y$ must be compactly generated. □

8.7.21 **Theorem**. *If* $p : X \longrightarrow X'$ *and* $q : Y \longrightarrow Y'$ *are identifications of compactly generated spaces, then* $p \times q : X\widetilde{\times}Y \longrightarrow X'\widetilde{\times}Y'$ *is an identification too.*

Proof: Since $p \times q$ factors as $(p \times \mathrm{id}) \circ (\mathrm{id} \times q)$ and the composition of identifications is again an identification, it is enough to prove the case $Y = Y'$ and $q = \mathrm{id}$.

Let $A \subset X'\widetilde{\times}Y$ be such that $(p \times \mathrm{id})^{-1}(A)$ is closed in $X\widetilde{\times}Y$ and take a compact set $C \subset X'\widetilde{\times}Y$. Let moreover D and E be the images of C in X' and Y under the projections, respectively. Then $D\widetilde{\times}E$ is compact. It will be enough to prove that $A \cap (D\widetilde{\times}E)$ is closed, since in that case also $A \cap C$ will be closed and since $X'\widetilde{\times}Y$ is compactly generated, A will be closed too. This will prove that $p \times \mathrm{id}$ is an identification.

Since $(p \times \mathrm{id})^{-1}(D \widetilde{\times} E) = p^{-1}(D) \widetilde{\times} E$ is closed in $X \widetilde{\times} Y$, we have that $(p \times \mathrm{id})^{-1}(A \cap (D \widetilde{\times} E))$ is closed in $p^{-1}(D) \widetilde{\times} E$. Replacing X, X', and Y by $p^{-1}(D)$, D and E, respectively, we reduce the proof to the case in which X' and Y are compact spaces. Thus by 8.7.20, $X' \widetilde{\times} Y = X' \times Y$ and $X \widetilde{\times} Y = X \times Y$.

Hence let us assume that X' and Y compact and let $W \subset X' \times Y$ be such that $(p \times \mathrm{id})^{-1}(W)$ is open in $X \times Y$. Take $(x_0', y_0) \in W$. Take $x_0 \in X$ such that $p(x_0) = x_0'$. Since (x_0, y_0) lies in the open set $(p \times \mathrm{id})^{-1}(W)$ and since Y is compact, there exists a neighborhood V of y_0 such that $\{x_0\} \times \overline{V} \subset (p \times \mathrm{id})^{-1}(W)$. Take $U = \{x \in X \mid \{p(x)\} \times \overline{V} \subset W\}$. Let us see that U is open in X. If $x_1 \in U$, then we can cover $\{x_1\} \times \overline{V}$ by products of open sets contained in $(p \times \mathrm{id})^{-1}(W)$ and and choose from them a finite subcover. Then the intersection of the factors in X of this finite number of products is a neighborhood N of x_1 such that $N \times \overline{V} \subset (p \times \mathrm{id})^{-1}(W)$. Therefore U is open. By definition, U is saturated with respect to p, namely $U = p^{-1}p(U)$. Hence, since p is an identification, $p(U)$ is open in X'. Moreover, since $(x_0', y_0) \in p(U) \times V$ and $p(U) \times V$ is open and lies in W, then W itself is open, as we wanted to prove. □

8.7.22 **Lemma**. *If X and Y are Hausdorff spaces, then the two topological spaces $c(X) \widetilde{\times} c(Y)$ and $c(X \times Y)$ coincide.*

Proof: Since the identity maps $c(X) \longrightarrow X$ and $c(Y) \longrightarrow Y$ are continuous, then the identity map $i : c(X) \times c(Y) \longrightarrow X \times Y$ is also continuous. Hence each compact set in $c(X) \times c(Y)$ is also compact in $X \times Y$. Let now A be a compact set in $X \times Y$. Since its projections B and C in X and Y, respectively, are compact, then so they are in $c(X)$ and $c(Y)$. Thus $B \times C$ is compact in $c(X) \times c(Y)$. Consequently $i|_{B \times C}$ is a homeomorphism. Since $A \subset B \times C$, we have that A is compact in $c(X) \times c(Y)$ and, because $c(X) \times c(Y)$ and $X \times Y$ have the same compact sets, by definition of the construction k (8.7.13) the compactly generated topologies associated to both coincide. □

If $X \subset Y$ and Y is compactly generated, it may happen that X with the relative topology is not compactly generated (see Example 8.7.5).

8.7.23 Definition. Let Y be compactly generated and $X \subset Y$. The *compactly generated relative topology* on X is the compactly generated topology associated to the usual relative topology, i.e. the *compactly generated subspace* is $c(X)$. A map $e : X \longrightarrow Y$ between compactly generated spaces is a *compactly generated embedding* if f is a homeomorphism of X onto the compactly generated subspace $c(f(X))$.

A compactly generated embedding $e : X \longrightarrow Y$ is characterized by a universal property analogous to the one of the induced topology (4.1.1). Namely, if $f : Z \longrightarrow Y$ is a map between compactly generated spaces such that $f(Z) \subset f(X)$, then there is a unique map $f' : Z \longrightarrow X$ such $e \circ f' = f$.

8.7.24 Exercise. Prove that the universal property stated above characterizes the compactly generated embeddings.

The following result, dual to 8.7.21, is immediate.

8.7.25 **Theorem**. *If $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are compactly generated embeddings, then $f \times g : X \tilde{\times} Y \longrightarrow X' \tilde{\times} Y'$ is also an embedding.* □

We shall now relate the compactly generated topology with the topology of function spaces that we discussed in Section 8.5.

Let X and Y be Hausdorff spaces and take the $\mathbf{M}(X, Y)$ space of maps from X to Y with the compact-open topology. Again, even if X and Y are compactly generated, $\mathbf{M}(X, Y)$ is not necessarily compactly generated. For instance, if X is discrete with dos points, then $\mathbf{M}(X, Y) = Y \times Y$. As we already mentioned, there is an example of Dowker's that shows that this space may not be compactly generated.

8.7.26 Definition. Let X and Y be Hausdorff spaces. Define $Y^X = c(\mathbf{M}(X, Y))$.

(We are abusing the notation. One should not confuse Y^X with the subbasic sets Q^K of the compact-open topology in $\mathbf{M}(X, Y)$.)

If we restrict to the class of compactly generated spaces, X^Y has the desired property. We have the following.

8.7.27 **Theorem**. *Let X and Y be compactly generated spaces. Then the evaluation map*

$$e : Y^X \tilde{\times} X \longrightarrow Y$$

is continuous.

The proof is based on the following lemma.

8.7.28 **Lemma**. *Let X and Y be Hausdorff spaces. Then the evaluation map*

$$e : \mathbf{M}(X, Y) \times X \longrightarrow Y$$

is continuous on compact sets.

Proof: Since every compact set in the product lies inside the product of its images under the projections, it is enough to prove that e is continuous on any compact set of the form $F \times A$, where $F \subset \mathbf{M}(X,Y)$ and $A \subset X$ are compact. Take $(f_0, x_0) \in F \times A$ and let $V \subset Y$ be open such that $f_0(x_0) \in V$. Since f_0 is continuous, there is a neighborhood U of x_0 in X for which $f_0(\overline{U}) \subset V$. Hence $(\overline{U}, V) \cap F)\widetilde{\times}U)$ is open in $F\widetilde{\times}A$, contains (f_0, x_0) and under e it is mapped into V. This proves that e is continuous on compact sets. □

Proof of 8.7.27: Applying the construction k to

$$e : \mathbf{M}(X,Y) \times X \longrightarrow Y$$

we obtain a continuous map

$$e : c(\mathbf{M}(X,Y) \times X) \longrightarrow c(Y)\,.$$

And if X and Y are compactly generated, this map is the same as

$$Y^X \widetilde{\times} X \longrightarrow Y\,,$$

since $c(\mathbf{M}(X,Y) \times X) = c(\mathbf{M}(X,Y))\widetilde{\times}c(X) = Y^X \widetilde{\times} X$. □

The notation Y^X makes sense for compactly generated spaces, since the exponential laws hold, namely

$$(Y\widetilde{\times}Z)^X = Y^X \widetilde{\times} Z^X$$
$$Z^{X\widetilde{\times}Y} = (Z^Y)^X\,,$$

whose proof we shall omit, since a version of the second one was already given in 8.6.2 (for details, see [4], [5], and [18]).

8.8 k-Spaces

We finish this chapter introducing an alternative class of convenient topological spaces devised by R. M. Vogt [19]. This is a broader class than that of compactly generated spaces, since it does not require the assumption that the spaces be Hausdorff. Indeed, this class inlcudes the class of compactly generated spaces. We only shall sketch the highlights of Vogt's class and recommend the reader to develop all proofs in detail.

8.8.1 Definition. A *k-space* X is a topological space with the property that a set $C \subseteq X$ is closed if and only if $\alpha^{-1}C \subseteq K$ is closed for any continuous map

$\alpha : K \longrightarrow X$, where K is any compact Hausdorff space. There is a similar construction that associates to every topological space X a k-space $k(X)$ with the same underlying set and a finer topology as just defined. Thus the identity $k(X) \longrightarrow X$ is continuous.

8.8.2 Exercise. Show that a compactly generated space X is a k-space.

8.8.3 Exercise. Prove that the construction $k(X)$ is functorial, i.e. it has the same properties of $c(X)$ given in the subsequent paragraphs to 8.7.13.

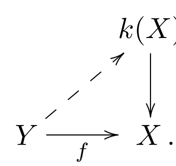
8.8.4 Exercise. Reformulate the properties given for $c(X)$ in 8.7.16 (except (b)) for the case of $k(X)$ and prove them.

8.8.5 Exercise. Prove that the construction $k(X)$ has a universal property that characterizes it, namely

(a) The identity map $k(X) \longrightarrow X$ is continuous.

(b) If Y is a k-space and $f : Y \longrightarrow X$ is continuous, then $f : Y \longrightarrow k(X)$ is continuous.

In other words the following is a commutative diagram of continuous maps:



8.8.6 Exercise. Show that if X is a k-space, then a map $f : X \longrightarrow Y$ is continuous if and only if the composite $f \circ \alpha : K \longrightarrow Y$ is continuous for every map $\alpha : K \longrightarrow X$, where K is any compact Hausdorff space.

8.8.7 Exercise. Let X be a k-space. Show the following:

(a) If $A \subseteq X$ is closed, then A with the usual relative topology is a k-space.

(b) If $A \subseteq X$ is arbitrary, then A with the usual relative topology is not in general a k-space. Give A the k-topology associated to the usual relative topology. Call this the *relative k-topology* and denote it by $k(A)$. Show:

   (i) If A is closed, then $k(A) = A$.

   (ii) The inclusion $i_A : k(A) \hookrightarrow X$ is continuous.

(iii) Given a k-space Y and $f : Y \longrightarrow A$, then $f : Y \longrightarrow k(A)$ is continuous if and only if $i_A \circ f : Y \longrightarrow X$ is continuous (cf. 4.1.4(ii)).

As in the case of the class of compactly generated spaces, instead of the usual topological product, we take its image under the construction k. Namely, we define a product $X \widehat{\times} Y = k(X \times Y)$.

One may prove the following two useful properties (see [19] and cf. 8.7.12, as well as 8.7.21):

1. If X is a k-space and $p : X \longrightarrow X'$ is an identification, then X' is a k-space; and
2. if $p : X \longrightarrow X'$ and $q : Y \longrightarrow Y'$ are identifications between k-spaces, then $p \widehat{\times} q : X \widehat{\times} Y \longrightarrow X' \widehat{\times} Y'$ is an identification.

One may also formulate and prove the corresponding results for function spaces, like the exponential laws.

Chapter 9 Other separability axioms

In this chapter we shall study separability properties for sets rather than for points. First we shall see the notion of normal space, and using it we shall prove the famous Tietze extension theorem. Further we shall analyze the concept of completely regular spaces, which will be useful to construct another compactification, namely the Stone–Čech compactification. We shall be able to prove that the members of an important class of topological spaces, namely the completely regular and second-countable spaces, are metrizable.

The last concept that we shall study in this chapter will be paracompact spaces, which is very important in analysis and several branches of topology.

9.1 Normal spaces

Normality is a separability property of sets. Metric spaces have this property, but they are the only ones. This property has two important results as a consequence, namely Urysohn's lemma and the Tietze extension theorem, which will be proved in this section.

9.1.1 Definition. Let X be a topological space and take $A \subset X$. We say that $U \subset X$ is a *neighborhood* of A in X if U is a neighborhood of each point $a \in A$. In other words, U is neighborhood of A if and only if $A \subset U^\circ$.

9.1.2 **Theorem**. *Let X be a topological space. The following are equivalent:*

(R) *The closed neighborhoods of each $x \in X$ form a neighborhood basis.*

(R′) *Given a closed set $B \subset X$ and a point $x \in X - B$, there are disjoint neighborhoods U of B and V of x.*

(R″) *Given a closed set $B \subset X$ and a compact set $K \subset X - B$, there are disjoint neighborhoods U of B and V of K.*

Proof:

(R″) ⇒ (R′) This is obvious, since any point x is compact.

(R′) ⇒ (R) Let U be a neighborhood of $x \in X$. We have to show that there is a closed neighborhood U' of x contained in U. For this, we assume that U is open (otherwise we take its interior) and take $B = X - U$, which is closed. Since $x \in X - B$, by (R') there are neighborhoods V of x and W of B such that $V \cap W = \emptyset$. We may assume that W is open. Hence $x \in V \subset X - W \subset X - B = U$, i.e. $X - W$ is a closed neighborhood of x which is contained in the given neighborhood U.

(R) ⇒ (R″) Let $B \subset X$ be closed and take $K \subset X - B$. Since $X - B$ is open, it is a neighborhood of K and therefore also of x for all $x \in K$. Let U_x be an open neighborhood of x such that $\overline{U}_x \subset X - B$. Since K is compact, $K \subset U_{x_1} \cup \cdots \cup U_{x_k} = U$ for finitely many points $x_1, \ldots, x_k \in K$. Let $V = X - (\overline{U}_{x_1} \cup \cdots \cup \overline{U}_{x_k}) \supset B$. Clearly U and V are the desired disjoint neighborhoods. □

9.1.3 Exercise. Prove directly (R′) ⇒ (R″).

In 8.2.18 we defined the concept of regular space as a topological space which satisfies (R). Theorem 9.1.2 gives us equivalent axioms for regularity.

Axiom (R″) indicates the possibility to separate by neighborhoods a closed set and a compact set. It is not always possible to separate two closed sets in a regular space.

9.1.4 Definition. Let X be a topological space. We say that X is *normal* if X satisfies

(N) Let $A, B \subset X$ be disjoint closed sets. Then there are disjoint neighborhoods U of A and V of B.

A normal space which is also Hausdorff is also called a T_4-*space*.

If X is a Hausdorff space, then clearly (N) ⇒ (R″). Indeed, we have the next result.

9.1.5 **Proposition**. *If X is a regular T_1-space, then X is a Hausdorff space.* □

9.1.6 Definition. A regular T_1-space is called T_3.

Hence $(T_3) \Rightarrow (T_2) \Rightarrow (T_1)$.

9.1.7 REMARK. Some authors define a regular space as we define T_3-space, namely asking that the given space is also T_1 (or T_2). Also a space is frequently called normal if it satisfies (N) and is Hausdorff.

9.1.8 **Proposition**. *Let X be a Hausdorff space and take disjoint compact sets $K, L \subset X$. Then there are disjoint neighborhoods U of K and V of L.*

Proof: Let us first assume that $L = \{y\}$. Since X is Hausdorff, then for all $x \in K$ there are neighborhoods U_x of x and V_x of y such that $U_x \cap V_x = \emptyset$. On the other hand, since K is compact, there are $x_1, \dots, x_k \in K$ such that $K \subset U_{x_1} \cup \cdots \cup U_{x_k} = U$. Take $V = V_{x_1} \cap \cdots \cap V_{x_k}$, which is also a neighborhood of y. Clearly U and V are the desired neighborhoods.

To see the general case, for each $y \in L$ take neighborhoods U_y of K and V_y of y such that $U_y \cap V_y = \emptyset$. Since L is compact, there are $y_1, \dots, y_l \in L$ such that $L \subset V_{y_1} \cup \cdots \cup V_{y_l} = V$. Put $U = U_{y_1} \cap \cdots \cap U_{y_l}$ which is a neighborhood of K too. Clearly U and V are now the desired neighborhoods. □

Since every compact set in a Hausdorff space is closed and every closed set in a compact space is compact, the following theorem is immediate.

9.1.9 **Theorem**. *Every compact Hausdorff space is a T_4-space; in particular it is normal.* □

Normality is a hereditary property with respect to closed sets.

9.1.10 **Proposition**. *Every closed subspace of a normal space is normal. Thus every closed subspace of a T_4-space is a T_4-space.* □

9.1.11 EXERCISE.

(a) Give an example of a normal space with a subspace which is not normal.

(b) Show with an example that not every product of normal spaces is normal.

We already observed that every T_4-space is regular. However, when the space is not Hausdorff, axiom (N) does not imply axiom (R).

9.1.12 EXAMPLE. Let X be the Sierpinski space consisting of two points x, y and only one nontrivial open set, namely $\{x\}$. Clearly X satisfies (N) but not (R').

The most natural example of T_4-spaces is given by the metric spaces. Namely, let X be a metric space and take two nonempty subsets $A, B \subset X$. We define their *distance* by

$$\mu(A,B) = \text{ínf}\{d(a,b) \mid (a,b) \in A \times B\}\,, \tag{9.1.13}$$

where d is the metric in X. It is clear that if $A \cap B \neq \emptyset$, then $\mu(A,B) = 0$. The converse is false, as the next example shows.

9.1.14 Example. Let $X = \mathbb{R}^2$ have the usual metric and take $A = \{(x,0) \mid x \in \mathbb{R}\}$ and $B = \{(x, 1/x) \mid x > 0\}$. The sets A and B are closed and disjoint, however their distance is $\mu(A,B) = 0$, as one easily checks.

The next result is clear.

9.1.15 **Proposition**. *If A is nonempty, then $\mu(x,A) = 0$ if and only if $x \in \overline{A}$.* □

9.1.16 **Theorem**. *Sea X un metric space with metric d and take a nonempty subset $A \subset X$. Then the function $\delta_A : X \longrightarrow \mathbb{R}$ given by $\delta_A(x) = \mu(x,A)$ is continuous.*

Proof: For $x, y \in X$, $z \in A$, the triangle inequality tells us

$$d(x,z) \leq d(x,y) + d(y,z)\,.$$

Therefore, for each $z \in A$, one has

$$\mu(x,A) \leq d(x,y) + d(y,z)\,,$$

so that

$$\mu(x,A) \leq d(x,y) + \mu(y,A)\,.$$

This shows the inequality

$$|\delta_A(x) - \delta_A(y)| = |\mu(x,A) - \mu(y,A)| \leq d(x,y)\,,$$

which clearly implies the continuity of δ_A. □

From 9.1.15 and 9.1.16, we obtain the next statement.

9.1.17 **Proposition**. *Take a nonempty set $A \subset X$, then $\overline{A} = \cap_{n=1}^{\infty} Q_n$, where $Q_n = \{x \in X \mid \mu(x,A) < 1/2^n\}$, which is an open set. In particular, every closed set is a countable intersection of open sets and, consequently, every open set is a countable union of closed sets.* □

Since δ_A is a continuous real-valued function, we obtain the next result from Corollary 8.1.32.

9.1.18 **Corollary**. *Take a nonempty closed set $A \subset X$ and a nonempty compact set $B \subset X$. If $A \cap B = \emptyset$, then $\mu(A, B) > 0$.* □

The main theorem of this section is the following.

9.1.19 **Theorem**. *Every metric space is a* T_4*-space.*

Proof: Let $A, B \subset X$ be two nonemepty disjoint closed sets and let $h : X \longrightarrow \mathbb{R}$ be given by $h(x) = \mu(x, A) - \mu(x, B) = \delta_A(x) - \delta_B(x)$. Clearly, h is a continuous function so that we have open sets

$$U = \{x \mid h(x) < 0\} \quad \text{and} \quad V = \{x \mid h(x) > 0\}\,,$$

which are obviously disjoint. Obviously, if $x \in A$, then $\mu(x, B) > 0$ (otherwise $x \in \overline{B} = B$), so that $h(x) < 0$, namely $A \subset U$. Analogously $B \subset V$. This shows that it is possible to separate A and B by neighborhoods. □

9.1.20 Note. Every subspace of a metric space is metric and therefore a T_4-space.

9.1.21 Definition. A normal space X such that every subspace of X is normal, is called *completely normal*.

Therefore, every metric space is completely normal.

9.1.22 **Lemma**. *Axiom* (N) *is equivalent to*

(N') *For every closed subset $B \subset X$ and every open subset $Q \subset X$ such that $B \subset Q$, there is an open set $U \subset X$ such that*

$$B \subset U \subset \overline{U} \subset Q\,.$$

One such open set U is called a *shrinking* of the neighborhood Q de B.

Proof:

(N) $\Longrightarrow$ (N') Let B be closed in X and let Q be an open neighborhood of B. Then B and $X - Q$ are disjoint closed sets and by (N) there are disjoint open neighborhoods U of B and V of $X - Q$, namely $U \subset X - V$. The set $X - V$ is closed. Therefore, $B \subset U \subset \overline{U} \subset X - V \subset X - (X - Q) = Q$.

(N') $\Longrightarrow$ (N) Take disjoint open sets $A, B \subset X$. Therefore, $Q = X - A$ is an open neighborhood of B. By (N') there is an open neighborhood U of B such that $\overline{U} \subset Q$. Consequently, $X - \overline{U} \supset X - Q = A$, and U and $X - \overline{U}$ are disjoint neighborhoods of B and A, respectively. □

The repeated application of the previous lemma produces a very interesting construction. Namely, if we take disjoint closed sets A and B in a space X that satisfies (N), then take

$$U(1) = X - A\,,$$

which is an open neighborhood of B. Applying the lemma, there is an open neighborhood $U(0)$ of B such that

$$\overline{U(0)} \subset U(1)\,.$$

If we continue this process, we obtain open sets $U(\frac{1}{2}), U(\frac{1}{4}), U(\frac{3}{4})$ such that

$$\overline{U(0)} \subset U(\tfrac{1}{4}) \subset \overline{U(\tfrac{1}{4})} \subset U(\tfrac{1}{2}) \subset \overline{U(\tfrac{1}{2})} \subset U(\tfrac{3}{4}) \subset \overline{U(\tfrac{3}{4})} \subset U(1)\,.$$

Continuing this process of putting a new neighborhood between every two already obtained, we get inductively, for each $k/2^n$, $k = 0, \ldots, 2^n$, open sets $U(\frac{k}{2^n})$ such that

$$\text{(7.1.23)} \qquad \overline{U(\tfrac{k}{2^n})} \subset U(\tfrac{k+1}{2^n}) \qquad k = 0, \ldots, 2^n - 1\,,$$

since given $U(\frac{k}{2^n})$ and applying the lemma to (7.1.23), we obtain $U(\frac{2k+1}{2^{n+1}})$ such that

$$\overline{U(\tfrac{k}{2^n})} \subset U(\tfrac{2k+1}{2^{n+1}}) \subset \overline{U(\tfrac{2k+1}{2^{n+1}})} \subset U(\tfrac{k+1}{2^n})\,.$$

This is the family of neighborhoods that corresponds to the stage $n+1$.

Thus we have obtained for all $r \in [0,1]$ such that $r = \frac{k}{2^n}$, open sets $U(r)$ such that

$$r < r' \Rightarrow \overline{U(r)} \subset U(r')\,.$$

Now, for any $t \in [0,1]$ we define the open set

$$U(t) = \bigcup_{r \leq t} U(r)$$

and one has

$$\text{(9.1.24)} \qquad t < t' \Rightarrow \overline{U(t)} \subset U(t')\,,$$

since if $t < t'$, then there are k and n such that $t < \frac{k}{2^n} < \frac{k+1}{2^n} < t'$. Hence

$$U(t) \subset U(\tfrac{k}{2^n}) \subset \overline{U(\tfrac{k}{2^n})} \subset U(\tfrac{k+1}{2^n}) \subset U(t')\,.$$

Moreover, we can extend our definition by putting $U(t) = \emptyset$ if $t < 0$ and $U(t) = X$ if $t > 1$. Then (9.1.24) is still valid.

Define $f(x) = \inf\{t \in \mathbb{R} \mid x \in U(t)\}$.

Since $U(t) = X$ for $t > 1$, then $f(x) \leq 1$ if $x \in X$, and since $U(t) = \emptyset$ for $t < 0$, then $f(x) \geq 0$ for all $x \in X$.

Moreover, if $B \neq \emptyset$, then $f(x) = 0$ for all $x \in B$ and if $A \neq \emptyset$, then $f(x) = 1$ for all $x \in A$.

We shall now show that the function f is continuous. Take $x_0 \in X$ and $\varepsilon > 0$. Put $t_0 = f(x_0)$. We shall prove that there is a neighborhood of x_0 such that $|f(x) - t_0| \leq \varepsilon$ for all x in this neighborhood.

If $x \in U(t_0 + \varepsilon)$, then $f(x) \leq t_0 + \varepsilon$. Moreover, if $x \in X - \overline{U(t_0 - \varepsilon)}$, then $f(x) \geq t_0 - \varepsilon$ (since if $f(x) < t_0 - \varepsilon$, then $x \in U(t_0 - \varepsilon) \subset \overline{U(t_0 - \varepsilon)}$). Thus if $x \in V = U(t_0 + \varepsilon) \cap (X - \overline{U(t_0 - \varepsilon)})$, then $|f(x) - t_0| \leq \varepsilon$. Also V is open and $x_0 \in V$ since $f(x_0) = t_0$. Therefore $x_0 \in U(t_0 + \varepsilon)$ y $x_0 \notin U(t_0 - \frac{\varepsilon}{2}) \supset \overline{U(t_0 - \varepsilon)}$.

As a consequence of all the previous, we have that f is continuous. Thus, given disjoint closed sets A and B in X, we have constructed a continuous function $f : X \longrightarrow [0,1]$ such that $f|_B = 0$ y $f|_A = 1$.

Conversely, given such a function f, we have that the sets $f^{-1}[0,1/2)$ and $f^{-1}(1/2,1]$ are open and disjoint and they contain B and A, respectively. Thus we have proved the following.

9.1.25 **Theorem**. (Urysohn's lemma) *In a topological space X axiom* (N) *(of Definition 9.1.4) and*

(N″) *Given disjoint closed sets $A, B \subset X$, there is a continuous function $f : X \longrightarrow [0,1]$ such that $f|_A \equiv 1$ and $f|_B \equiv 0$.* □

9.1.26 Definition. A function $f : X \longrightarrow [0,1]$, as in (N″), is called *Urysohn's function* and an open neighborhood of $A \subset X$ (or an open set) of the form $f^{-1}(0,1]$ is called *numerable neighborhood* of A (*numerable open set*).

9.1.27 Note. If $B = \emptyset$, then we can take $f \equiv 1$.

9.1.28 Note. If A and B are disjoint closed sets in X and we consider $Y = A \cup B$, then the function defined on $Y \subset X$ with value 1 on A and 0 on B is continuous. The statement of (N″) means that this function can be extended to all of X.

9.1.29 **Theorem**. (Tietze extension theorem) *In a topological space X axiom* (N) *is equivalent to*

(N‴) *If G is a closed set in X and $g : G \longrightarrow [a,b]$ is a continuous function, then G can be continuously extended to a function $f : X \longrightarrow [a,b]$.*

To prove (N‴) one constructs a sequence of functions $f_n : X \longrightarrow [a,b]$ such that the restrictions $f_n|_G$ approximate more and more the function g and the sequence $\{f_n(x)\}$ converges uniformly.

The induction step to construct f_{n+1} once we have f_n is a consequence of the next lemma.

9.1.30 **Lemma**. *Let X be a normal space (that satisfies* (N)*) and take a closed set G in X and $b > 0$. If $u : G \longrightarrow [-b, b]$ is continuous, then there is $v : X \longrightarrow [-\frac{b}{3}, \frac{b}{3}]$ such that for all $x \in G$,*

$$|u(x) - v(x)| \leq \frac{b}{3}\,.$$

Proof: Take

$$A = \{x \in G \mid u(x) \leq -\frac{b}{3}\} \quad \text{and} \quad B = \{x \in G \mid u(x) \geq \frac{b}{3}\}.$$

The sets A and B are closed and disjoint, so that there is a continuous function $w : X \longrightarrow [0, 1]$ such that $w|_A = 0$ and $w|_B = 1$. Define

$$v(x) = \frac{2b}{3}w(x) - \frac{b}{3}\,.$$

Then for all x, $|v(x)| \leq \frac{b}{3}$ and if $x \in G$, then

$$|u(x) - v(x)| = |u(x) + \frac{b}{3} - \frac{2b}{3}w(x)|.$$

Take $x \in G$. There are three possibilities, namely

(i) $u(x) \leq -\dfrac{b}{3}$.

In this case $x \in A$, so that $v(x) = -\dfrac{b}{3}$. Therefore,

$$-\frac{2b}{3} = -b + \frac{b}{3} \leq u(x) - v(x) \leq -\frac{b}{3} + \frac{b}{3} = 0\,.$$

(ii) $u(x) \geq \frac{b}{3}$.

In this case $x \in B$, so that $v(x) = \frac{b}{3}$. Therefore

$$\frac{2b}{3} = b - \frac{b}{3} \geq u(x) - v(x) \geq \frac{b}{3} - \frac{b}{3} = 0\,.$$

(iii) $-\dfrac{b}{3} < u(x) < \dfrac{b}{3}$.

In this case $-\frac{b}{3} \leq v(x) \leq \frac{b}{3}$, so that

$$-\frac{2b}{3} = -\frac{b}{3} - \frac{b}{3} < u(x) - v(x) < \frac{b}{3} + \frac{b}{3} = \frac{2b}{3}.$$

In all three cases one has the desired inequality. □

Proof de 9.1.29:

$(N''') \Rightarrow (N'')$ is clear, and since $(N'') \Leftrightarrow (N)$, we have that $(N''') \Rightarrow (N)$.

The only assertion that we have to prove is $(N) \Rightarrow (N''')$.

Without loss of generality, we may assume that $a = -1$ and $b = 1$, namely that we have a function $g : G \longrightarrow [-1, 1]$.

By the previous lemma there is a continuous function $f_1 : X \longrightarrow [-\frac{1}{3}, \frac{1}{3}]$ such that

$$|g(x) - f_1(x)| \leq \frac{2}{3}\,.$$

Take $g_1 = g - f_1 : G \longrightarrow [-\frac{2}{3}, \frac{2}{3}]$. Therefore, there is a continuous function $v_1 : X \longrightarrow [-\frac{2}{9}, \frac{2}{9}]$ such that

$$|g_1(x) - v_1(x)| \leq (\frac{2}{3})^2\,.$$

Take $f_2 = f_1 + v_1 : X \longrightarrow [-1 + (\frac{2}{3})^2, 1 - (\frac{2}{3})^2]$.

Assume that we have constructed a continuous function

$$f_n : X \longrightarrow [-1 + (\frac{2}{3})^n, 1 - (\frac{2}{3})^n]$$

such that $|g(x) - f_n(x)| \leq (\frac{2}{3})^n$ for $x \in G$. If we apply the previous lemma to $g_n = g - f_n|_G$, we have that there is another continuous function

$$v_n : X \longrightarrow [-\frac{1}{3}(\frac{2}{3})^n, \frac{1}{3}(\frac{2}{3})^n]$$

such that $|g_n(x) - v_n(x)| < (\frac{2}{3})^{n+1}$ for $x \in G$.

Now we define $f_{n+1}(x) = f_n(x) + v_n(x)$. Clearly $|g(x) - f_{n+1}(x)| < (\frac{2}{3})^{n+1}$ for $x \in G$ and

$$f_{n+1} : X \longrightarrow [-1 + (\frac{2}{3})^{n+1}, 1 - (\frac{2}{3})^{n+1}]\,.$$

Since $|g(x) - f_n(x)| < (\frac{2}{3})^n$ for all n, we get for $x \in G$ that $\lim f_n(x) = g(x)$.

Take an arbitrary point $x \in X$ and $m \geq n$. Then

$$|f_m(x) - f_n(x)| = |\sum_{k=n}^{m-1} v_k(x)| < \sum_{k=n}^{\infty} \frac{1}{3}(\frac{2}{3})^k\,,$$

where this latter term is the tail of a convergent series. Therefore, taking n large enough, we can make this term as small as we want. Therefore, for all $x \in X$, the sequence $f_n(x)$ converges to the value $f(x)$. The function

$$f : X \longrightarrow [-1, 1]$$

is continuous, since the convergence is *uniform*, as we shall now see. Let n be sufficiently large so that $|f_m(x) - f_n(x)| < \varepsilon$ for all $m \geq n$. Hence $|f(x) - f_n(x)| \leq \varepsilon$.

Since f_n is continuous at each $x_0 \in X$, there is a neighborhood U of x_0 such that for $x \in U$, we have

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < 3\varepsilon\,.$$

This shows that f is continuous. Thus the theorem is proved. □

9.1.31 **Corollary**. *Let X be a normal space (which satisfies* (N)*) and take a closed set $G \subset X$. If $g : G \longrightarrow [a, b)$ is a continuous function, then g admits a continuous extension $f : X \longrightarrow [a, b)$. Moreover, if $g : G \longrightarrow (a, b)$ is a continuous function, then g admits a continuous extension $f : X \longrightarrow (a, b)$.*

Proof: To prove the first case, we may assume without loss of generality that that $[a, b) = [0, 1)$. By Theorem 9.1.29, there is a continuous extension $h : X \longrightarrow [0, 1]$. Put $B = h^{-1}\{1\}$, which is a closed set. B and G are disjoint. By Urysohn's lemma 9.1.25 there is a continuous function $v : X \longrightarrow [0, 1]$ such that $v|_G = 1$ and $v|_B = 0$. Define

$$f(x) = v(x) \cdot h(x)\,.$$

Then f is a continuous extension of g such that $f(x) < 1$, since one would only have $f(x) = 1$ if $h(x) = 1$, but this only happens for $x \in B$, and in this case $v(x) = 0$.

For the second case, we may assume without loss of generality that that $(a, b) = (-1, 1)$ and we take

$$g^+(x) = \max\{g(x), 0\} \quad \text{and} \quad g^-(x) = \max\{-g(x), 0\}\,,$$

so that $g(x) = g^+(x) - g^-(x)$. By definition, the functions g^+ and g^- are such that

$$g^+, g^- : G \longrightarrow [0, 1)\,.$$

Hence, in the first case, there are continuous functions $f^+, f^- : X \longrightarrow [0, 1)$ which extend g^+ and g^-, respectively. Then the desired extension is $f = f^+ - f^-$. □

As a consequence of the previous results we obtain the following.

9.1.32 **Theorem**. (Tietze–Urysohn extension theorem) *Let X be a normal space and take a closed set $A \subset X$. Consider a family of intervals $\{I_\lambda\}_{\lambda\in\Lambda}$ in $\mathbb{R}$ and take the product $Y = \prod_\lambda I_\lambda$. Then every continuous map $g : A \longrightarrow Y$ admits a continuous extension $f : X \longrightarrow Y$. (In particular, Y can be taken as $\mathbb{R}^n$.)* □

9.1.33 Remark. The proof of the previous theorem requires only that X satisfies (N), namely, it need not be Hausdorff. However, the assumption that it is also Hausdorff is the most usual.

9.1.34 Exercise. A topological space E is called a *Euclidean neighborhood retract* or briefly, an ENR, if there is an embedding $i : E \hookrightarrow \mathbb{R}^n$ such that $i(E)$ has an open neighborhood $U \subset \mathbb{R}^n$ and there is a retraction $r : U \longrightarrow E$, i.e. $r \circ i = \mathrm{id}_E$. Show that if X satisfies (N), $G \subset X$ is a closed set, and $g : G \longrightarrow E$ es continuous, then there is an open set $V \subset X$ such that $G \subset V$, and there is an extension $f : V \longrightarrow E$ de g.

9.2 Completely regular spaces

In this section we shall study another important separability property, namely the complete regularity. As it is the case with normality, this property is shared by a large class of topological spaces. The complete regularity is an essential property for constructing another important compactification, the so-called Stone–Čech compactification that we shall study in the next section.

We start with the next definition, to state it, we observe first that if X is a normal topological space, then the following holds:

(CR) For all $x \in X$ and every neighborhood U of x there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and the restriction $f|_{(X-U)} = 1$.

To prove this, notice that since X is, in particular, a Hausdorff space, then $G = \{x\}$ is closed. On the other hand, the set $F = X - U^\circ$ is also closed and $F \cap G = \emptyset$. Since X is normal, there is a continuous function $f : X \longrightarrow I$ such that $f|_G = 0$ and $f|_F = 1$, as we wanted to show.

9.2.1 Definition. A topological space X is *completely regular* if X satisfies (CR). Moreover, we say that it is a $\mathrm{T}_{3\frac{1}{2}}$-space if it is Hausdorff too.

Indeed, we have the next result.

9.2.2 **Theorem**. *Every* T_4*-space* X *is a* $\mathrm{T}_{3\frac{1}{2}}$*-space.* □

The concept of regularity is not stronger than that of complete regularity, as the next result shows.

9.2.3 **Theorem**. *Every* $\mathrm{T}_{3\frac{1}{2}}$*-space* X *is a* T_3*-space.*

Proof: Let X satisfy (CR) and take $x \in X$ and a neighborhood U of x. We wish to show that there is a closed neighborhood V of x contained in U.

By (CR) we know that there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f|_{(X-U)} = 1$. If we define $V = f^{-1}[0, 1/2]$, then clearly V is a closed neighborhood of x, as we wished. □

9.2.4 Note. Indeed we have that (CR) ⇒ (R), without requiring the assumption that the space is Hausdorff. However it is not true that (N) ⇒ (CR), since we require that a point is closed. Hence every completely regular space X is a regular space.

An example of a regular space which is not completely regular is not simple. We refer the reader to [20, 18G] to see one.

Differing from axiom (N), property (CR) is inherited by any subspace. It is from this property that the adverb "completely" comes. To see that property (CR) is hereditary, take $A \subset X$ and $x \in A$. A neighborhood of x in A is the intersection of a neighborhood U of x in X with A. By assumption, we know that there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f|(X - U) \equiv 1$. The restriction $f|_A$ is then a separating function of x and the complement of $A \cap U$ in A. We thus have the next result.

9.2.5 **Theorem**. *If X is completely regular, then also each of it subspaces is completely regular. In particular, if X is a $\mathrm{T}_{3\frac{1}{2}}$-space, then also each of its subspaces is a $\mathrm{T}_{3\frac{1}{2}}$-space.* □

In 9.1.9 we proved that each compact Hausdorff space is normal. Therefore, we have the following.

9.2.6 **Corollary**. *Every compact Hausdorff space is a $\mathrm{T}_{3\frac{1}{2}}$-space.* □

We also saw in 9.1.19 that every metric space is T_4-space. Therefore, we also have the following.

9.2.7 **Corollary**. *Every metric space is a $\mathrm{T}_{3\frac{1}{2}}$-space (in particular, it is completely regular).* □

Every Hausdorff locally compact space X is (an open) subspace of its Alexandroff compactification X^*, which is a compact Hausdorff space. By 9.2.6, X^* is a $\mathrm{T}_{3\frac{1}{2}}$-space. Hence, by 9.2.5, we have the following.

9.2.8 **Proposition**. *Every Hausdorff locally compact space X is a $T_{3\frac{1}{2}}$-space (in particular, it is completely regular).* □

With respect to the relationship between property (CR) and products, we have the next result.

9.2.9 **Theorem**. *Let $\{X_\lambda \mid \lambda \in \Lambda\}$ Be a nonempty family of nonempty topological spaces. Their product $X = \prod_{\lambda\in\Lambda} X_\lambda$ ia a is completely regular space if and only if X_λ is a completely regular space for all $\lambda \in \Lambda$. In particular, X is a $T_{3\frac{1}{2}}$-space if and only if X_λ is a $T_{3\frac{1}{2}}$-space for all $\lambda \in \Lambda$.*

Proof: Assume that X_λ satisfies (CR) for all $\lambda \in \Lambda$ and take $x \in X$, $x = (x_\lambda)$. Let U be a neighborhood of x in X. U contains a product neighborhood Q of the form $Q = \prod Q_\lambda$, where Q_λ is a neighborhood x_λ in X_λ for all λ and $Q_\lambda = X_\lambda$ for all but finitely many λ, say for $\lambda = \kappa_1, \dots, \kappa_m$, $Q_\lambda \neq X_\lambda$. Take

$$f_i : X_{\kappa_i} \longrightarrow I$$

given by $f_i(x_\lambda) = 0$ and $f_i|_{(X_{\kappa_i} - Q_{\kappa_i})} = 1$. Then the composite

$$X \xrightarrow{p_i} X_{\kappa_i} \xrightarrow{f_i} I\,,$$

where p_i is the projection, is continuous, so that $f : X \longrightarrow I$ given by

$$f(y) = \max\{f_i p_i(y) = f_i(y_{\kappa_i}) \mid i = 1, \dots, m\}\,,$$

is continuous too, where $y = (y_\lambda)_{\lambda\in\Lambda}$ is an arbitrary point in X. Clearly $f(x) = 0$. Moreover, if $y \notin U$, then $y_{\kappa_i} \notin Q_{\kappa_i}$ for some $i = 1, \dots, m$. Therefore, $f_i(y_{\kappa_i}) = 1$ for that i. Hnece $f(y) = 1$.

Conversely, since $X = X_\kappa \times \prod_{\lambda\neq\kappa} X_\lambda$, it is enough to assume that X is a product of two factors, say $X = X_1 \times X_2$, and to prove that if X satisfies (CR), then X_1 satisfies (CR) too.

This is clear since, if $x_2 \in X_2$, then X_1 is homeomorphic to $X_1 \times \{x_2\}$, which is a subspace of X. Therefore, if X satisfies (CR), then X_1 does too. □

9.2.10 Definition. Consider $I = [0, 1]$ and let Λ be a set of indexes. Take the product I^Λ of copies of I, one for each $\lambda \in \Lambda$. We call I^Λ a Λ-*cube*. If $\Lambda = \{1, \dots, n\}$, then $I^\Lambda = I^n$ is the n-*cube* and if $\Lambda = \mathbb{N}$, then $I^\Lambda = I^\omega$ is the *Hilbert cube*.

Since the product of Hausdorff spaces is Hausdorff, the Tychonoff theorem provides us with the following.

9.2.11 **Proposition**. *Every Λ-cube is a compact Hausdorff space, hence it is a* T_4-*space (normal) and in particular, a* $\mathrm{T}_{3\frac{1}{2}}$-*space (completely regular).*

Let X be a topological space, take $\Phi = \{\varphi : X \longrightarrow I \mid \varphi \text{ is continuous}\}$ and the map

$$j : X \longrightarrow I^{\Phi}\,,$$

given by $j(x) = (\varphi(x))_{\varphi\in\Phi}$. Since $p_\varphi \circ j = \varphi$ is continuous, where p_φ is the corresponding projection, then j es continuous. Property (N) suggests to consider the numerable open sets $W_\varphi = \varphi^{-1}[0,1)$ and $W'_\varphi = p_\varphi^{-1}[0,1)$. Since $[0,1)$ is open in I, W_φ and W'_φ are open in X and I^Φ, respectively, and one has that $W_\varphi = j^{-1}W'_\varphi$. Consequently

$$j(W_\varphi) = W'_\varphi \cap j(X)\,. \tag{9.2.12}$$

If X satisfies (CR), then the sets W_φ are sufficient open sets in X, in the sense of the following lemma.

9.2.13 **Lemma**. *Take $\Phi = \{\varphi : X \longrightarrow I \mid \varphi$ is continuous$\}$. The space X satisfies* (CR) *if and only if $\{W_\varphi \mid \varphi \in \Phi\}$ is a basis for the topology of X.*

Proof: Take a nonempty open set $Q \subset X$ and take $x \in Q$. Let $\varphi : X \longrightarrow I$ be such that $\varphi(x) = 0$ and $\varphi(X - Q) = 1$. We have that φ is such that $x \in W_\varphi \subset Q$.

Conversely assume that $\{W_\varphi\}$ is a basis for the topology of X and take a point $x \in X$ and a neighborhood U of x. By assumption, there is a continuous function $\varphi : X \longrightarrow [0,1]$ such that $x \in W_\varphi \subset U$. Hence $\varphi(x) < 1$ and $\varphi(y) = 1$ for $y \in X - U$. Sea $u : X \longrightarrow [0,1]$ given by

$$u(y) = \max\{0, \frac{\varphi(y) - \varphi(x)}{1 - \varphi(x)}\}\,,$$

which is a continuous function such that $u(x) = 0$ and $u(y) = 1$ and $y \in X - U$. □

Let X be a $\mathrm{T}_{3\frac{1}{2}}$-space (completely regular and Hausdorff) and let $\Gamma \subset \Phi$ be a set of continuous functions $\varphi : X \longrightarrow I$. We shall say that Γ is *admissible* if the family $\{W_\varphi \mid \varphi \in \Gamma\}$ is a basis for the topology of X. Let $j_\Gamma : X \longrightarrow I^\Gamma$ be defined as above. The restricted map $j' : X \longrightarrow j_\Gamma(X)$ is also continuous and (9.2.12) states that $j(W_\varphi)$ is an open set in $j_\Gamma(X)$. Therefore j' is an open surjective map.

Let Γ be admissible. Since X is Hausdorff, given $x \neq y$ in X, there are disjoint open sets U and V in X such that $x \in U$ and $y \in V$. Therefore, there is $\varphi \in \Gamma$ such that $x \in W_\varphi \subset U$, so that $\varphi(x) < 1$, while $\varphi(y) = 1$. Hence $j_\Gamma(x) \neq j_\Gamma(y)$.

We have the following.

9.2.14 **Lemma**. *Let X be a $T_{3\frac{1}{2}}$-space (completely regular and Hausdorff) and take an admissible set Γ of continuous functions. Then X es homeomorphic to a subspace of I^Γ.*

Proof: The map j_Γ is continuous and injective. Since by (9.2.12) j' is open, one has that $j' : X \longrightarrow j_\Gamma(X) \subset I^\Gamma$ is a homeomorphism. □

Conversely, if X is homeomorphic to a subspace of a compact Hausdorff space, then X is a $\mathrm{T}_{3\frac{1}{2}}$-space (completely regular and Hausdorff), since every compact Hausdorff space is a $\mathrm{T}_{3\frac{1}{2}}$-space (completely regular and Hausdorff). Thus this property is hereditary.

Hence we have the next result.

9.2.15 **Theorem**. (Tychonoff) *A topological space is a $T_{3\frac{1}{2}}$-space (completely regular and Hausdorff) if and only if es homeomorphic to a subspace of a compact Hausdorff space. More specifically, X is a $T_{3\frac{1}{2}}$-space (completely regular and Hausdorff) if and only if the map $j_\Gamma : X \longrightarrow I^\Gamma$ is an embedding for some set Γ of continuous functions $\varphi : X \longrightarrow I$.* □

9.3 The Stone–Čech compactification

Starting with all work done in the previous section, we can now give a new way to compactify spaces, if they are $\mathrm{T}_{3\frac{1}{2}}$-spaces. Namely, we shall introduce the so-called Stone–Čech compactification.

Let $j : X \longrightarrow I^\Phi$, $j(x) = (\varphi(x))_{\varphi\in\Phi}$ be the embedding of a $\mathrm{T}_{3\frac{1}{2}}$-space X into the Φ-cube, where $\Phi = \{\varphi : X \longrightarrow I \mid \varphi \text{ es continuous}\}$. Take

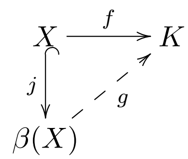
$$\beta(X) = \overline{j(X)} \subset I^\Phi .$$

Since X is homeomorphic to $j(X)$, we have that X can be embedded as a dense subspace of $\beta(X)$, where $\beta(X)$ is a compact Hausdorff space. In other words, $\beta(X)$ is a compactification of X.

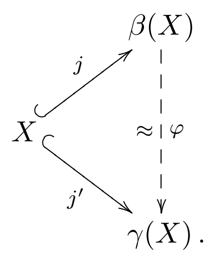
9.3.1 Definition. Let X be a $\mathrm{T}_{3\frac{1}{2}}$-space (completely regular and Hausdorff). The embedding $j : X \longrightarrow \beta(X)$ is called the *Stone–Čech compactification* of X. Abusing we call $\beta(X)$ with the same name.

9.3.2 **Theorem**. *Let X be a $T_{3\frac{1}{2}}$-space. Then its Stone–Čech compactification is characterized by the following universal property:*

(SČ) For every compact Hausdorff space K and every continuous map $f : X \longrightarrow K$, there is a unique continuous map $g : \beta(X) \longrightarrow K$ such that $g \circ j = f$, namely, such that the following diagram commutes*:*

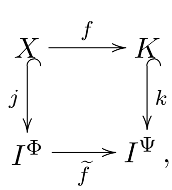


Proof: The property characterizes uniquely $j : X \longrightarrow \beta(X)$, since if $j' : X \longrightarrow \gamma(X)$ has the same property, we may assume $K = \gamma(X)$ and $f = j'$. By (SČ), there is (a unique) $\varphi : \beta(X) \longrightarrow \gamma(X)$ such that $\varphi \circ j = j'$. Analogously, by the corresponding property of j', there is (a unique) $\psi : \gamma(X) \longrightarrow \beta(X)$ such that $\psi \circ j' = j$. Therefore $\psi \circ \varphi \circ j = j$ and $\varphi \circ \psi \circ j' = j'$. The uniqueness part of property (SČ) implies that $\psi \circ \varphi = \mathrm{id}_{\beta(X)}$ and $\varphi \circ \psi = \mathrm{id}_{\gamma(X)}$. In other words, $\beta(X)$ and $\gamma(X)$ are homeomorphic and the homeomorphism transforms j into j'. In a commutative diagram, we have



We must proof that $j : X \longrightarrow \beta(X)$ has property (SČ). Hence take a compact Hausdorff space K and a continuous map $f : X \longrightarrow K$. Consider $\Phi = \{\varphi : X \longrightarrow I \mid \varphi \text{ is continuous}\}$ and $\Psi = \{\psi : K \longrightarrow I \mid \psi \text{ is continuous}\}$. Moreover, let $j : X \longrightarrow I^\Phi$ and $k : K \longrightarrow I^\Psi$ be given by $j(x) = (\varphi(x))_{\varphi\in\Phi}$ and $k(y) = (\psi(y))_{\psi\in\Psi}$, so that $\beta(X) = \overline{j(X)} \subset I^\Phi$ and $\beta(K) = \overline{k(K)} \subset I^\Psi$. Since K is compact, $k(K)$ is compact too. Therefore $k(K) = \beta(K)$ and $k : K \longrightarrow \beta(K)$ is a homeomorphism.

Let $\widetilde{f} : I^\Phi \longrightarrow I^\Psi$ be given by $\widetilde{f}(t_\varphi) = (t_{\psi f})$, namely the ψ-component of $\widetilde{f}(t_\varphi)$ is $t_{\psi f}$. We have a commutative diagram



as one easily checks. Since j is a homeomorphism of X onto $j(X)$, we have that $\widetilde{f}(j(X)) \subset \beta(K)$. Therefore, by the continuity of $\widetilde{f}$, we have $\widetilde{f}(\beta(X)) = \widetilde{f}(\overline{j(X)}) \subset \overline{\widetilde{f}(j(X))} \subset \beta(K)$. Hence let $g : \beta(X) \longrightarrow K$ be such that $k \circ g = \widetilde{f}|_{\beta(X)} : \beta(X) \longrightarrow \beta(K)$. This map g has the desired property.

The uniqueness of g is an immediate consequence of the fact that X embeds into $\beta(X)$ as a dense subspace and of the fact that K is Hausdorff. □

From the previous proof, we obtain the following assertion.

9.3.3 **Proposition**. *Let K be a compact Hausdorff space, then its Stone–Čech compactification $\beta(K)$ is homeomorphic to K.* □

From Theorem 9.3.2 we get the following consequence.

9.3.4 **Corollary**. *The β-construction is functorial, namely, if X, Y, and Z are $T_{3\frac{1}{2}}$-spaces (completely regular and Hausdorff) and the maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous, then there are maps $\beta(f) : \beta(X) \longrightarrow \beta(Y)$ and $\beta(g) : \beta(Y) \longrightarrow \beta(Z)$ such that*

$$\beta(g \circ f) = \beta(g) \circ \beta(f)\,.$$

Moreover, $\beta(\mathrm{id}_X) = \mathrm{id}_{\beta(X)}$.

Proof: To construct, say, $\beta(f)$, take $K = \beta(Y)$ and the map $X \longrightarrow \beta(Y)$ induced by f. Then use axiom (SČ) to obtain $\beta(f)$. Define $\beta(g)$ analogously. An application of the uniqueness mentioned in (SČ) provides us with the desired formula. The same uniqueness guarantees that if we apply the β-construction to the identity we obtain the identity. □

Take a $\mathrm{T}_{3\frac{1}{2}}$-space (completely regular and Hausdorff) X and all its compactifications $e : X \longrightarrow K$, i.e. $e : X \longrightarrow e(X)$ is a homeomorphism such that $e(X)$ is dense in K. Consider the family

$$\mathcal{C} = \{(K, e) \mid e : X \longrightarrow K \text{ is a compactification of X}\}.$$

One can define a partial order in $\mathcal{C}$ as follows.

$$(K_1, e_1) \leq (K_2, e_2) \Leftrightarrow \exists f : K_2 \longrightarrow K_1 \text{ continuous such that } f \circ e_2 = e_1.$$

From 9.3.2 we obtain the following consequence.

9.3.5 **Corollary**. *The Stone–Čech compactification $(\beta(X), j)$ is the supreme of the family $\mathcal{C}$, i.e $(K, e) \leq (\beta(X), j)$ for every compactification $(K, e) \in \mathcal{C}$.*

9.3.6 Note. If $\Gamma \subset \Phi = \{f : X \longrightarrow I \mid f \text{ is continuous}\}$ is admissible, then the embedding $j : X \longrightarrow I^\Gamma$ defined in the previous section is such that its restriction $j' : X \longrightarrow \overline{j(X)}$ is a compactification. However, in general, this compactification does not coincide with the Stone–Čech compactification. This is th case of course too, if X is embedded into I^Γ for an arbitrary Γ. For instance, if $X = (0, 1]$, then $X \subset I$ and $\overline{X} = I$. In other words, the embedding $(0, 1] \hookrightarrow I$ is a compactification, but it is not the Stone–Čech compactification, since it does not have the universal property (SČ), as the map $f : (0, 1] \longrightarrow [-1, 1]$, given by $f(t) = \sin(1/t)$ shows, because it does not admit an extension to $I = [0, 1]$. More generally, the closed ball $\mathbb{B}^n$ is not the Stone–Čech compactification of the open ball $\overset{\circ}{\mathbb{B}}{}^n$.

9.3.7 Exercise. Show that the real projective space $\mathbb{RP}^n$ is a compactification of $\mathbb{R}^n$, which is different from the Alexandroff and the Stone–Čech compactifications. (*Hint:* Remember that the real projective space $\mathbb{RP}^n$ can be defined as the quotient $\mathbb{B}^n/\!\sim$ such that $\sim$ identifies antipodal points of the boundary $\mathbb{S}^{n-1}$ of the ball. This way, one has an inclusion of the open ball, which is homeomorphic to the Euclidean space $\mathbb{R}^n$, into the projective space. For the second part of the exercise, show that that inclusion, namely $\mathbb{R}^n \hookrightarrow \mathbb{B}^n \longrightarrow \mathbb{RP}^n$, is indeed a compactification, but does not have the universal proeprty (SČ).)

9.3.8 Exercise. Take $f \in \Phi$, namely a continuous function $f : X \longrightarrow I$. By the universal property (SČ), f has an extension to $\beta(X) \subset I^\Phi$. Show that this extension is nothing else but the restriction of the projection $\text{proj}_f : I^\Phi \longrightarrow I$.

9.3.9 **Theorem**. *A space X is a $T_{3\frac{1}{2}}$-space (completely regular and Hausdorff) and is second-countable if and only if X can be embedded as a subspace of the Hilbert-cube I^ω.*

Proof: First notice that I^ω is second-countable, since it is a countable product of intervals (which are second-countable), and a $\mathrm{T}_{3\frac{1}{2}}$-space. Since these properties are hereditary, we have that if X can be embedded as a subspace of the Hilbert-cube I^ω, then it is second-countable and a $\mathrm{T}_{3\frac{1}{2}}$-space.

Conversely, it is enough to show that there exists a countable subset Γ of Φ such that $\{W_\varphi \mid \varphi \in \Gamma\}$ is a basis of the topology of X. Since X is second-countable, this is an immediate consequence of Lemma 3.4.26.

If $\{W_{\varphi_n}\}$ is a countable basis contained in $\{W_\varphi \mid \varphi \in \Phi\}$, then $\Gamma = \{\varphi_n\}$ is a countable subset of Φ such that X can be embedded into I^Γ. □

The relationship between the Stone–Čech compactification and the Alexandroff compactification studied in the previous chapter is given by the next exercise.

9.3.10 EXERCISE. Let X be a $\mathrm{T}_{3\frac{1}{2}}$-space and $j : X \longrightarrow \beta(X)$ its Stone–Čech compactification. Prove that the composite

$$k : X \overset{j}{\hookrightarrow} \beta(X) \overset{q}{\twoheadrightarrow} \beta(X)/(\beta(X) - j(X)) \ ,$$

where q is the quotient map, is homeomorphic to the Alexandroff compactification. In particular, if X is compact, then $\beta(X) - j(X) = \emptyset$ and by definition, $\beta(X)/\emptyset = \beta(X) \sqcup \{\infty\} \approx X \sqcup \{\infty\} = X^+$.

9.3.11 EXERCISE. Let X be a $\mathrm{T}_{3\frac{1}{2}}$-space and let $j : X \longrightarrow \beta(X)$ be its Stone–Čech compactification $k : X \longrightarrow X^*$ its Alexandroff compactification. By Corollary 9.3.5, there is a continuous map $f : \beta(X) \longrightarrow X^*$ such that $f \circ j = k$. Show that f is an identification such that $f(\beta(X) - j(X)) = \{\infty\}$.

9.3.12 EXERCISE. Analogously to the fact that the Stone–Čech compactification is (by 9.3.5) the supreme of all compactifications, prove that the Alexandroff compactification is the infimum, i.e., if $e : X \longrightarrow K$ is a compactification, prove that the composite

$$k : X \overset{e}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} K/(K - e(X)) \ ,$$

where q is the quotient map, is homeomorphic to the Alexandroff compactification, in the sense that one has a commutative triangle

$$\begin{array}{ccccc} & & X & & \\ & \overset{e}{\swarrow} & & \overset{i}{\searrow} & \\ K & \underset{q}{\twoheadrightarrow} & K/(K - e(X)) & \underset{\approx}{\longrightarrow} & X^* . \end{array}$$

This generalizes 9.3.10.

9.4 METRIZABLE SPACES

We have seen along the text that spaces whose topology comes from a metric have particularly rich topological properties. A natural question is which of these properties are enough to decide that the topological structure comes from a metric. In this section we shall prove metrizability results.

9.4.1 DEFINITION. Let X be a topological space. We say that X *metrizable* if one can define a metric on X such that the topology determined by the metric coincides with the given topology.

In this section we shall give a criterion to know when a space is metrizable.

9.4.2 **Lemma**. *Let X be a metric space with metric d. Then the function $d' : X \times X \longrightarrow \mathbb{R}^+$ given by*

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}\,,$$

is a bounded metric which defines the same topology on X as d does.

Proof: The function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ given by $f(t) = \frac{t}{1+t}$ defines an increasing homeomorphism $\mathbb{R}^+ \longrightarrow [0,1)$. Thus $d' = f \circ d$ is a metric with the desired properties, as one easily verifies. □

9.4.3 Note. The bounded metric defined in 9.4.2 is not the only one which defines the same topology as d. It is an easy *exercise* to show that the function

$$d''(x,y) = \min\{1, d(x,y)\}$$

is also a a bounded metric which defines the same neighborhood system on X as d and therefore the same topology.

9.4.4 **Theorem**. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a countable family of metrizable spaces. Then $\prod X_n$ is a metrizable space.*

Proof: Take a metric d_n on X_n bounded by 1. Then the function $d : \prod X_n \times \prod X_n \longrightarrow \mathbb{R}^+$ given by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n)\,,$$

where $x = \{x_n\}$ and $y = \{y_n\}$ are elements in $\prod X_n$, yields a bounded metric on $\prod X_n$.

In order to see that d defines on $\prod X_n$ the product topology, notice first that a basic neighborhood of a point $x \in \prod X_n$ has the form $V = \prod B_{\varepsilon_n}(x_n)$, where $\varepsilon_n = 1$ for $n > k$, $k \in \mathbb{N}$.

Take such a neighborhood V and put $\varepsilon = \min\{\varepsilon_n/2^n \mid n = 1, \ldots, k\}$, which is a positive number. Thus, if $y \in \prod X_n$ is such that $d(x,y) < \varepsilon$, then we have

$$\begin{aligned}
\frac{1}{2^n} d_n(x_n, y_n) &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) \\
&= d(x,y) \\
&< \varepsilon = \min\{\varepsilon_n/2^n\} \\
&\leq \frac{\varepsilon_n}{2^n}.
\end{aligned}$$

Therefore $d_n(x_n, y_n) < \varepsilon_n$ for $n = 1, \ldots, k$. Moreover $d_n(x_n, y_n) \leq \varepsilon_n$ for $n > k$. Consequently $B_\varepsilon(x) \subset V$.

Conversely, take $\varepsilon > 0$. We wish to show that there is a V as above such that $V \subset B_\varepsilon(x)$. Take $k \in \mathbb{N}$ such that

$$\sum_{n>k} \frac{1}{2^n} < \frac{\varepsilon}{2},$$

and take $\varepsilon_n = 2^{n-1}\varepsilon/k$ for $n \leq k$ and $\varepsilon_n = 1$ for $n > k$. Hence, if $y \in V = \prod B_{\varepsilon_n}(x_n)$, then $d_n(x_n, y_n) < \varepsilon_n$ for all n. Thus

$$\begin{aligned} d(x,y) &= \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) \\ &\leq \sum_{n=1}^{k} \frac{1}{2^n} d_n(x_n, y_n) + \sum_{n>k} \frac{1}{2^n} \\ &\leq \sum_{n=1}^{k} \frac{1}{2^n} \varepsilon_n + \frac{\varepsilon}{2} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

namely $V \subset B_\varepsilon(x)$. □

9.4.5 Note. The use of $1/2^n$ in the previous proof is simply to make things easy. Namely, if we have a convergent series $\sum_{n=1}^{\infty} c_n$ such that $c_n > 0$, then the function

$$d(x,y) = \sum_{n=1}^{\infty} c_n d_n(x_n, y_n)$$

is a bounded metric on $\prod X_n$ which defines the product topology as in the proof of 9.4.4.

9.4.6 Exercise. Prove directly, simplifying the proof of 9.4.4, that taking arbitrary $c_n > 0$ the finite product of metrizable spaces is metrizable.

9.4.7 Exercise. Take the Hilbert cube $Q = I^\omega$. Prove that its topology is given by the following metric:

$$d(s,t) = \sum_{n=1}^{\infty} \frac{1}{2^n} |t_n - s_n|.$$

We are now in position to prove the next result.

9.4.8 **Theorem**. (Urysohn's metrizability) *If a topological space X is second countable, then the following are equivalent:*

(a) *X is a is a* $T_{3\frac{1}{2}}$*-space (completely regular and Hausdorff).*

(b) *X is a* T_4*-space (normal and Hausdorff).*

(c) *X is metrizable.*

Proof: Every metrizable space is a T_4-space. Therefore (c) $\Rightarrow$ (b). We also know that every T_4-space is a $T_{3\frac{1}{2}}$-space, thus (b) $\Rightarrow$ (a). Finally, by 9.3.9, every second-countable T_4-space can be embedded in the Hilbert-cube I^ω. Moreover, by 9.4.4 the Hilbert-cube is metrizable, then every $T_{3\frac{1}{2}}$-space is metrizable. Consequently (a) $\Rightarrow$ (c). □

9.5 Paracompact spaces

We saw that every compact Hausdorff space is locally compact and normal, namely these are concepts which generalize the compactness concept. The property of being compact is a global property. However the property of being locally compact is a local property. Compactness was defined using global finiteness properties of open covers of the given space. In this section we shall study locally finiteness properties of open covers of a topological space.

9.5.1 Definition. Let X be a topological space and take a family $\mathcal{C} = \{A_\lambda \mid \lambda \in \Lambda\}$ of subsets of X. We say that the family $\mathcal{C}$ is *locally finite* if every point $x \in X$ has a neighborhood V such that $V \cap A_\lambda \neq \emptyset$ only for finitely many values of λ.

9.5.2 **Lemma**. *Let* $\mathcal{C} = \{A_\lambda \mid \lambda \in \Lambda\}$ *be a locally finite family of subsets of a topological space* X*. Then also the family of closures* $\overline{\mathcal{C}} = \{\overline{A}_\lambda \mid \lambda \in \Lambda\}$ *is locally finite.*

Proof: Take $x \in X$ and let V be an open neighborhood of x such that $V \cap A_\lambda \neq \emptyset$ only for finitely many values of $\lambda \in \Lambda$. If $V \cap A_\lambda = \emptyset$, then $A_\lambda \subset X - V$, where $X - V$ is closed. Hence $\overline{A}_\lambda \subset X - V$. Consequently $V \cap \overline{A}_\lambda \neq \emptyset$, up to finitely many values of $\lambda \in \Lambda$. □

9.5.3 **Lemma**. *Let* $\{F_\lambda \mid \lambda \in \Lambda\}$ *be a locally finite family of closed sets in a topological space* X*. Then the set* $F = \bigcup F_\lambda$ *is is closed in* X*.*

Proof: We shall prove that $X - F$ is open. Take $x \in X - F$. x has a neighborhood V which meets only finitely many of the sets F_λ. If $V \cap F_\lambda = \emptyset$ for all λ, then $V \subset X - F$. If, on the contrary, $V \cap F_\lambda \neq \emptyset$ for $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $V \cap \cap_{i=1}^k (X - F_{l_i})$ is an open neighborhood of x which is contained in $X - F$. Therefore $X - F$ is open. □

9.5.4 Remark. If $\{F_\lambda\}$ is a *closed* cover, namely a cover whose elements are closed sets, which is locally finite in X and if $f : X \longrightarrow Y$ is a map, then f is continuous if and only if all restrictions $f|_{F_\lambda}$ are continuous. This can be analogously shown to the case of finite covers, making use of the previous lemma.

9.5.5 Definition. Let $\mathcal{C} = \{A_\lambda \mid \lambda \in \Lambda\}$ and $\mathcal{C}' = \{B_\mu \mid \mu \in \mathrm{M}\}$ be covers of the set X. We say that $\mathcal{C}'$ is *finer* than $\mathcal{C}$, or that it is a *refinement* of $\mathcal{C}$, if each set in $\mathcal{C}'$ is contained in at least one set in $\mathcal{C}$. In other words, if $\mathcal{C}'$ is finer than $\mathcal{C}$, then (using the axiom of choice) one can define a function $\Phi : \mathrm{M} \longrightarrow \Lambda$ such that for all $\mu \in \mathrm{M}$, one has $B_\mu \subset A_{\Phi(\mu)}$. One says that the refinement is *precise* if we can take $\Lambda = \mathrm{M}$ and $\Phi = \mathrm{id}$.

9.5.6 Remark. From the previous definition it is clear that a subcover $\mathcal{C}'$ of $\mathcal{C}$ is a refinement. It is also immediate that the relation of being finer is transitive. If, on the other hand, $\mathcal{C}_1 = \{A_\lambda \mid \lambda \in \Lambda\}$ and $\mathcal{C}_2 = \{B_\mu \mid \mu \in \mathrm{M}\}$ are covers of a space X, then the intersections $C_{\lambda\mu} = A_\lambda \cap B_\mu$, for all λ and μ form a new cover $\mathcal{C}$ of X which is a common refinement of $\mathcal{C}_1$ and $\mathcal{C}_2$. Obviously one can omit all empty intersections and if $\mathcal{C}_1$ and $\mathcal{C}_2$ are open covers, then also $\mathcal{C}$ is an open cover.

9.5.7 Definition. A topological space X is *paracompact* if it is Hausdorff and satisfies the following condition:

(PC) Every open cover of X admits a locally finite open refinement.

9.5.8 **Lemma**. *If a topological space X is paracompact, then every open cover of X admits a locally finite precise open refinement.*

Proof: Let $\mathcal{C} = \{A_\lambda\}_{\lambda \in \Lambda}$ be an open cover of X. Since X is paracompact, there is a refinement $\mathcal{C}' = \{B_\mu\}_{\mu \in M}$. Let $\Phi : M \longrightarrow \Lambda$ be a function such that $B_\mu \subset A_{\Phi(\mu)}$. Define $B'_\lambda = \bigcup_{\Phi(\mu)=\lambda} B_\mu$. Clearly the family $\mathcal{C}'' = \{B'_\lambda\}_{\lambda \in \Lambda}$ is an open refinement of $\mathcal{C}$ such that $B'_\lambda \subset A_\lambda$. Moreover, $\mathcal{C}''$ is a locally finite refinement, since for all $x \in X$ there is a neighborhood V of x such that V meets only finitely many elements $B_\mu \in \mathcal{C}'$. Since the elements $B'_\lambda \in \mathcal{C}''$ are unions of elements of $\mathcal{C}'$, then V meets only finitely many of them. □

Clearly one has the following result.

9.5.9 **Proposition**. *Every compact Hausdorff space is paracompact.* □

9.5.10 **Theorem**. *Every closed subspace of a paracompact space is paracompact.*

Proof: Let X be paracompact and take a closed subspace $Y \subset X$. Let $\mathcal{C} = \{A_\lambda \mid \lambda \in \Lambda\}$ be an open cover of Y. For each A_λ there is an A'_λ open in X such that $A'_\lambda \cap Y = A_\lambda$. The family $\mathcal{C}' = \{X - Y\} \cup \{A'_\lambda \mid \lambda \in \Lambda\}$ is an open cover of X and since X is paracompact, there is a locally finite refinement $\mathcal{B}' = \{B'_\mu \mid \mu \in M\}$. Then the family $\mathcal{B} = \{B'_\mu \cap Y \mid \mu \in M\}$ is a locally finite refinement of $\mathcal{C}$. □

We shall see in what follows what relation does the paracompactness concept have with the concepts of separability studied before. To do that we consider the following.

9.5.11 **Lemma**. *Let X be a paracompact space and take disjoint closed subsets F and G of X. If for each point $x \in F$ there are disjoint neighborhoods of x and G, then F and G have disjoint neighborhoods too.*

Proof: For each $x \in F$ take disjoint open neighborhoods U_x of x and V_x of G. Consider the open cover $\mathcal{C}$ of X consisting of the sets U_x and $X - F$. By Lemma 9.5.9, there is a locally finite refinement of $\mathcal{C}$ consisting of an open set $A \subset X - F$ and open sets $W_x \subset U_x$. Define

$$W = \bigcup_{x \in F} W_x, \quad H = \bigcup_{x \in F} \overline{W}_x \,.$$

Since $A \subset X - F$ and $A \cup W = X$, W is an open neighborhood of F. By Lemmas 9.5.2 and 9.5.3, H is closed. Moreover, by the choice of V_x one has that $\overline{W}_x \subset \overline{U}_x \subset X - V_x \subset X - G$, so that $H \subset X - G$, i.e. $X - H \supset G$. Hence $U = X - H$ is an open neighborhood of G, which does not meet the neighborhood W of F. □

We can now prove the following result.

9.5.12 **Theorem**. *Every paracompact space X is T_4 (in particular, normal).*

Proof: We have to prove that two disjoint closed sets F and G in X have disjoint (open) neighborhoods.

First take a closed set $F \subset X$ and $G_y = \{y\} \subset X - F$. Since X is Hausdorff, G_y is closed and hence, for each $x \in F$ there are disjoint (open) neighborhoods of x and G_y. Consequently, by the previous lemma, there are disjoint open neighborhoods U_y of F and V_y of G_y, namely of y.

Therefore, we see that the assumptions of the previous lemma are again fulfilled for F and $y \in G$. Thus F and G have disjoint (open) neighborhoods, and thus (N) holds and so X is normal. Since it is Hausdorff, it is T_4. □

For T_3-spaces, the definition of a paracompact space can be formulated in several equivalent ways, varying the type of sets with which one makes the refinements. We have the following result.

9.5.13 **Theorem**. (E. Michael) *Let X be a is* T_3 *(i.e. regular and Hausdorff). The following are equivalent:*

(a) *X is paracompact.*

(b) *Every open cover of X has an open refinement which is a union of an at most countable collection locally finite families of open sets.*

(c) *Every open cover of X has a locally finite refinement consisting of not necessarily open nor closed sets.*

(d) *Every open cover of X has a locally finite closed refinement.*

Proof:

(a) ⇒ (b) is clear.

(b) ⇒ (c) Let $\mathcal{C} = \{A_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X. By (b) there is an open refinement $\mathcal{C}' = \{B_{n,\mu}\}_{(n,\mu)\in\mathbb{N}\times M}$ such that for each fixed n, the family $\mathcal{C}'_n = \{B_{n,\mu}\}_{\mu\in M}$ is locally finite (but not necessarily a cover). For each n, take

$$B_n = \bigcup_{\mu\in M} B_{n,\mu}\,.$$

Then the family $\mathcal{C}'' = \{B_n\}_{n\in\mathbb{N}}$ is an open cover of X. For each $i = 1, 2, \ldots$, define $C_i = B_i - \bigcup_{j<i} B_j$. The family $\{C_i\}$ is a refinement of $\mathcal{C}''$. It is a cover, since for each $x \in X$, $x \in C_{n(x)}$, where $n(x)$ is the first n for which x belongs to B_n. It is also locally finite since the neighborhood $B_{n(x)}$ of x does not meet any C_i for $i > n(x)$. It results that $\mathcal{B} = \{C_n \cap B_{n,\mu}\}$ is a refinement of $\mathcal{C} = \{A_\lambda\}$. It is locally finite, since for each $x \in X$ there is a neighborhood that meets C_n for at most finitely many values of n, and for each such n, the point x has a neighborhood that meets at most finitely many $B_{n,\mu}$. Hence $\mathcal{B}$ is the required refinement.

(c) ⇒ (d) Let $\mathcal{C}$ be an open cover of X. To each $x \in X$ we associate an element $A_x \in \mathcal{C}$, defined such that $x \in A_x$. Since X is regular, there is an open set B_x such that

$$x \in B_x \subset \overline{B}_x \subset A_x\,.$$

The family $\{B_x\}_{x\in X}$ is an open cover. By (c), this family has a precise locally finite refinement $\mathcal{B} = \{B'_x\}$. Therefore $\mathcal{B}' = \{\overline{B'}_x\}$ is also a locally finite family, and since $\overline{B'}_x \subset \overline{B}_x \subset A_x$ for each x, then $\mathcal{B}'$ is the desired refinement.

(d) ⇒ (a) Let $\mathcal{C}$ be an open cover of X. Take a locally finite closed refinement $\mathcal{C}'$. Then $x \in X$ has a neighborhood V_x that meets only finitely many sets $B \in \mathcal{C}'$. The cover $\{V_x\}_{x\in X}$ admits also a locally finite closed refinement $\mathcal{C}''$. Since each $C \in \mathcal{C}''$ meets only finitely many elements $B \in \mathcal{C}'$, we can enlarge each B to an open set G_B such that the family $\{G_B\}$ is locally finite. Namely, take

$$G_B = X - \bigcup\{C \in \mathcal{C}'' \mid C \cap B = \emptyset\}\,.$$

Associating to each $B \in \mathcal{C}'$ a unique set $A_B \in \mathcal{C}$ which contains it, it is clear that the family $\mathcal{B} = \{G_B \cap A_B\}$ is a locally finite open refinement of $\mathcal{C}$. □

9.5.14 Definition. Let $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ be an open cover of a topological space X. We say that $\mathcal{V}$ is *shrinkable* provided an open cover $\mathcal{V}' = \{V'_\lambda\}_{\lambda\in\Lambda}$ exists with the property that $\overline{V}'_\lambda \subset V_\lambda$. Of course, $\mathcal{V}'$ is called a *shrinking* of $\mathcal{V}$.

The following lemma will be useful below.

9.5.15 **Lemma**. *Let X be normal. Then every locally finite open cover is shrinkable.*

Proof: Let $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ be a locally finite open cover of X. By the well-order axiom (see [9]) we can well-order the set Λ, and for instance assume that $\Lambda = \{1, 2, \dots, \lambda, \dots\}$. We shall construct $\mathcal{V}' = \{V'_\lambda\}_{\lambda\in\Lambda}$ by transfinite induction as follows. Take

$$F_1 = X - \bigcup_{\lambda>1} V_\lambda\,.$$

Then $F_1 \subset V_1$, and since X is normal, there is an open set V'_1 such that

$$F_1 \subset V'_1 \subset \overline{V}'_1 \subset V_1\,.$$

Assume that F_μ and V'_μ have already been defined for all $\mu < \lambda$ and take

$$F_\lambda = X - [(\bigcup_{\mu<\lambda} V'_\mu) \cup (\bigcup_{\nu>\lambda} V_\nu)]\,.$$

Then F_λ is a closed set and $F_\lambda \subset V_\lambda$. Therefore we can find an open set V'_λ such that

$$F_\lambda \subset V'_\lambda \subset \overline{V}'_\lambda \subset V_\lambda\,.$$

Take $\mathcal{V}' = \{V'_\lambda \mid \lambda \in \Lambda\}$, which is a shrinking of $\mathcal{V}$, as far as it is a cover. To see this, take $x \in X$. Since $\mathcal{V}$ is locally finite, x belongs only to finitely many elements of $\mathcal{V}$, say that x belongs to $V_{l_1}, \dots, V_{l_k}$. Take $\lambda = \max\{\lambda_1, \dots, \lambda_k\}$. Therefore $x \notin V_\nu$ for $\nu > \lambda$ and thus, if $x \notin V'_\mu$ for $\mu < \lambda$, then $x \in F_\lambda \subset V'_\lambda$. This way, $x \in V'_\mu$ for some $\mu \leq \lambda$. Hence we have that $\mathcal{V}'$ is an open cover of X and therefore it is a shrinking of $\mathcal{V}$. □

9.5.16 EXERCISE. A cover of a topological space X is *pointwise finite* if each point of X belongs to at most finitely many elements of the cover. Following the proof of Lemma 9.5.15, show that X is normal if and only if every pointwise finite open cover of X is shrinkable.

In general, the product of paracompact spaces need not be paracompact.

9.5.17 EXERCISE. Let X be the following space. As a set take $X = (-1, 1]$ and let its topology have as open sets the intervals $(a, b]$. Show that every open cover of X admits a refinement consisting of a countable collection of open sets. (*Hint:* Construct the refinement from right to left.) Show that $X \times X$ is completely regular, however it is not normal. (*Hint:* Consider the set of rational numbers and the set of irrational points on the line $y = -x$ in $X \times X$.) Prove that X is first-countable and paracompact, but it is not metrizable.

However, we have the following results.

9.5.18 **Theorem**. *The product of a paracompact space and a compact space is a paracompact space.*

Proof: Let X be a paracompact space and K a compact space, and let $\mathcal{U}$ be an open cover of $X \times K$. For a fixed $x \in X$, there are finitely many elements in $\mathcal{U}$, say $U_1^x, \dots, U_{k_x}^x$, which cover $\{x\} \times K$. Take an open neighborhood V_x of x in X such that $V_x \times K \subset \bigcup_{i=1}^{k_x} U_i^x$, which exists because K is compact. The sets V_x, $x \in X$, form an open cover of X. Since X is paracompact, this cover admits a locally finite refinement $\mathcal{V}$. For each $V \in \mathcal{V}$, $V \subset V_x$ for some $x \in X$. Consider the sets $(V \times K) \cap U_i^x$, $i = 1, \dots, k_x$, built varying V in $\mathcal{V}$. This is a refinement of $\mathcal{U}$ and an open cover of $\mathcal{W}$ of $X \times K$. Moreover, given $(x, y) \in X \times K$, there is a neighborhood U of x which meets only finitely many elements $V \in \mathcal{V}$, so that the neighborhood $U \times K$ of (x, y) can only meet finitely many elements of $\mathcal{W}$. □

9.5.19 **Theorem**. *Every metric space X is paracompact.*

Proof: Since X is metric, it is T_3 (regular and Hausdorff, see 9.2.7). Let d be a metric in X. We shall see that X satisfies 9.5.13(b). Thus let $\mathcal{C} = \{A_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X and, as in the proof of 9.5.15, give Λ a well-order. For each $\lambda \in \Lambda$ and all $n \in \mathbb{N}$ define the following sets:

$$U_{n\lambda} = \{x \in X \mid \mu(x, X - A_\lambda) > \frac{1}{2^n}\},$$

$$T_{n\lambda} = U_{n\lambda} \cap (\cap_{\kappa<\lambda}(X - A_\kappa)), \qquad \lambda \in \Lambda,$$

$$B_{n\lambda} = \{x \in X \mid \mu(x, T_{n\lambda}) > \frac{1}{2^{n+2}}\},$$

where μ represents the distance between sets associated to d, as in (9.1.13). From 9.1.15, one can deduce that

$$\bigcup_n U_{n\lambda} = A_\lambda,$$

since $X - A_\lambda$ is a closed set. Hence A_λ is a union of countably many open sets. For a fixed n, the sets $T_{n\lambda}$ constitute a family of sets, which are not very close to each other, namely

$$\mu(T_{n\lambda}, T_{n\kappa}) \geq \frac{1}{2^n}, \qquad \lambda, \kappa \in \Lambda, \lambda \neq \kappa,$$

since if $x \in T_{n\lambda}$, $y \in T_{n\kappa}$ and $l < \kappa$, then $y \notin A_\lambda$ by definition of $T_{n\kappa}$. Therefore, from the definition of $U_{n\lambda}$, $d(x, y) > 1/2^n$.

Even when $T_{n\lambda}$ need not be open, $B_{n\lambda}$ is open. If the open sets $B_{n\lambda}$ and $B_{n\kappa}$ are nonempty, then, applying the triangle inequality, one obtains the inequality $\mu(B_{n\lambda}, B_{n\kappa}) \geq 1/2^{n+1}$. From here we can deduce that for any $x \in X$, any ball with center x and radius $1/2^{n+2}$ meets at most one set $B_{n\lambda}$. Hence $\mathcal{C}'_n = \{B_{n\lambda}\}_{\lambda\in\Lambda}$ is a locally finite family of open sets such that, by definition, $B_{n\lambda} \subset A_\lambda$.

To be able to apply 9.1.14(b), it is thus enough to verify that the family $\mathcal{C}' = \bigcup_n \mathcal{C}'_n = \{B_{n\lambda}\}$ is a cover. Take $x \in X$ and let $\lambda \in \Lambda$ be the minimal element such that $x \in A_\lambda$. This element exists due to the well-order of Λ. Taking n sufficiently large, one has $x \in U_{n\lambda}$. Since λ is minimal, $x \in T_{n\lambda}$ and hence $x \in B_{n\lambda}$, as desired. □

It follows from this result, among other consequences, that the Euclidean space $\mathbb{R}^n$ *is paracompact.*

Paracompactness allows us, in a sense, to reduce certain global properties to local properties. This happens, for instance, for continuity. To see it, consider the following.

9.5.20 DEFINITION. Let X be a topological space and let $f : X \longrightarrow \mathbb{R}$ be continuous. The *support* of f is the closed set

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}\,.$$

A way of describing the support of f is by saying that its complement $X -$ supp(f) is the maximal open set on which the function f is identically zero.

9.5.21 DEFINITION. Let $\{f_\lambda : X \longrightarrow \mathbb{R}\}_{\lambda\in\Lambda}$ be a family of continuous functions. The family is called a *partition of unity* if

(a) $f_\lambda(x) \geq 0$ for all $x \in X$.

(b) The collection of supports $\{\operatorname{supp}(f_\lambda)\}_{\lambda\in\Lambda}$ is locally finite, namely, for all $x \in X$ a neighborhood V of x exists such that $V \cap \operatorname{sop}(f_\lambda) \neq \emptyset$ only for finitely many elements $\lambda \in \Lambda$. In this case, $\sum_{\lambda\in\Lambda} f_\lambda(x)$ is well defined, since it is a finite sum, and it is continuous since the sum remains finite inside a neighborhood of each point.

(c) $\sum_{\lambda\in\Lambda} f_\lambda(x) = 1$ for all $x \in X$.

Thus the family of supports $\{\operatorname{supp}(f_\lambda)\}$ is a locally finite closed cover of the space X.

9.5.22 DEFINITION. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of the topological space X. We say that a partition of unity $\{f_\lambda\}_{\lambda\in\Lambda}$ en X *is subordinate* to the cover if for every $\lambda \in \Lambda$, $\operatorname{supp}(f_\lambda) \subset U_\lambda$.

9.5.23 **Theorem**. *Let X be a Hausdorff space. Then X is paracompact if and only if every open cover of X has a subordinate partition of unity.*

Proof: Take any open cover $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ of X and assume that X admits a subordinate partition of unity $\{f_\lambda\}_{\lambda\in\Lambda}$. Take $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$, where $V_\lambda = \{x \in X \mid f_\lambda(x) > 0\}$. It is clear that $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}$.

Conversely, let X be paracompact and let $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X. Assume that $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ is a precise locally finite refinement of $\mathcal{U}$. By 9.5.12, X is normal, and by 9.5.15, $\mathcal{V}$ is shrinkable, namely, there is an open cover $\mathcal{V}' = \{V'_\lambda\}_{\lambda\in\Lambda}$ such that $\overline{V}'_\lambda \subset V_\lambda$ for all $\lambda \in \Lambda$. The open cover $\mathcal{V}'$ is also shrinkable, so that there is another open cover $\mathcal{V}'' = \{V''_\lambda\}_{\lambda\in\Lambda}$ such that $\overline{V}''_\lambda \subset V'_\lambda$. Let $g_\lambda : X \longrightarrow I$ be such that $g_\lambda|_{\overline{V}''_\lambda} = 1$ and $g_\lambda|_{X-V'_\lambda} = 0$. Such a function exists because X is normal.

Since $\mathcal{V}''$ is a cover, $x \in X$ belongs to some V''_λ. Hence, for this λ, $g_\lambda(x) \neq 0$. Thus, since the cover $\mathcal{V}''$ is locally finite, the function $G : X \longrightarrow \mathbb{R}^+$ given by

$$G(x) = \sum_{\lambda\in\Lambda} g_\lambda(x)$$

is well defined, is continuous and is different from zero. Besides, $\operatorname{supp}(g_\lambda) \subset U_\lambda$. Therefore,the functions

$$f_\lambda = g_\lambda/G : x \mapsto g_\lambda(x)/G(x)\,, \quad \lambda \in \Lambda\,,$$

are also defined and they clearly constitute a partition of unity subordinate to the given cover. □

To finish this section, we shall show an interesting application of partitions of unity. For that we recall first that if $X \subset \mathbb{R}^n$, then we say that a function $f : X \longrightarrow \mathbb{R}$ is *smooth*, if it can be extended to an open set $U \subset \mathbb{R}^n$ such that $X \subset U$, and the extension has partial derivatives of all orders.

Consider the next assertion.

9.5.24 **Proposition**. *Let $f_\lambda : X \longrightarrow \mathbb{R}$, $\lambda \in \Lambda$, be a family of continuous [smooth] functions such that the family of supports $\{\operatorname{supp}(f_\lambda)\}$ is locally finite. Then the sum of functions $f = \sum_{\lambda\in\Lambda} f_\lambda$ is well defined and is continuous [smooth].*

Proof: Since $\{\operatorname{supp}(f_\lambda)\}$ is locally finite, one has that for each $x \in X$, the sum $\sum_{\lambda\in\Lambda} f_\lambda(x)$ is finite. Moreover there is an open neighborhood U of x such that $U_x \cap \operatorname{supp}(f_\lambda) \neq \emptyset$ only for finitely many indexes λ. Thus the function $f_U = f|_U = \sum_{\lambda\in\Lambda}(f_\lambda|_U)$ is a finite sum, and thus it is continuous [smooth]. This way, function f is continuous [smooth] on the open sets of a cover of X, so that, by 4.1.25, it must be continuous [smooth, since each of its partial derivatives is continuous on each open set]. □

9.5.25 NOTE. Let X be a topological space and let $\{U_\lambda\}$ be an open cover of X. Take a continuous [smooth] function $g_\lambda : U_\lambda \longrightarrow \mathbb{R}$ and assume that we can find a partition of unity $\{\pi_\lambda : X \longrightarrow \mathbb{R}\}$ [which is *smooth*, namely, such that each function is smooth] subordinate to the cover. Take $x \in X$ and define

$$f_\lambda(x) = \begin{cases} \pi_\lambda(x) g_\lambda(x) & \text{si } x \in U_\lambda \\ 0 & \text{si } x \in X - \operatorname{supp}(\pi_l)\,. \end{cases}$$

Clearly f_λ is continuous [smooth] and therewith the sum $\sum_{\lambda\in\Lambda} f_\lambda$ is continuous [smooth] too. This sum is usually denoted by

$$\sum_{\lambda\in\Lambda} \pi_\lambda g_\lambda\,.$$

and is called the *assembly* of the functions g_λ with the partition of unity.

We shall see now how the existence of smooth partitions of unity can be used to find smooth approximations of continuous functions. For that, if $X = \mathbb{R}^n$, it is possible to prove Theorem 9.5.23 to obtain a smooth partition of unity. Before we do it, we need the following smooth version of Urysohn's lemma (9.1.25) for $\mathbb{R}^n$, which will be crucial.

9.5.26 **Lemma**. *Let $A, B \subset \mathbb{R}^n$ be disjoint closed sets. Then a smooth function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ exists, such that $0 \leq g(x) \leq 1$, $g|_A = 1$ and $g|_B = 0$.*

Proof: We split the proof into two cases.

Case 1. Assume $A = B_\varepsilon(x_0)$, $B = \mathbb{R}^n - B^\circ_\delta(x_0)$, $0 < \varepsilon < \delta$.

Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$\alpha(t) = \begin{cases} \mathrm{e}^{-1/t^2} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0\,. \end{cases}$$

It is immediate to check that α is a smooth function.

Let now $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$\beta(t) = \frac{\alpha(1-t)}{\alpha(1-t) + \alpha(t)}\,.$$

This function is well defined, smooth and satisfies

$$\begin{cases} \beta(t) = 1 & \text{if } x \leq 0 \\ 0 < \beta(t) < 1 & \text{if } 0 < t < 1 \\ \beta(t) = 0 & \text{if } x \geq 1\,. \end{cases}$$

We can use it to define $g_0 : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$g_0(x) = \beta(\frac{|x - x_0|^2 - \varepsilon^2}{\delta^2 - \varepsilon^2})\,,$$

which is smooth and satisfies $g_0|_{B_\varepsilon(x_0)} = 1$, $g_0|_{\mathbb{R}^n - B^\circ_\delta(x_0)} = 0$.

Case 2. (General) Consider the open sets $V_1 = X - A$, $V_2 = X - B$ and take a countable family of closed balls $B_{\varepsilon_j}(x_j) \subset B_{\delta_j}(x_j)$, $0 < \varepsilon_j < \delta_j$, $j = 1, 2, \ldots$, such that

(a) each of the balls $B_{\delta_j}(x_j)$ lies inside V_1 or inside V_2, and

(b) the open balls $B^\circ_{\varepsilon_j}(x_j)$ cover $\mathbb{R}^n$.

One can indeed find such a family, since $\mathbb{R}^n$ is second-countable and thus we can find a countable family of balls that form a basis of the topology.

As in case 1, we have smooth functions $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $0 \leq g_j(x) \leq 1$ and

$$g_j|_{B_{\varepsilon_j}(x_j)} = 1 \qquad g_j|_{\mathbb{R}^n - B^\circ_{\delta_j}(x_j)} = 0\,.$$

Let now $h_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $h_j(x) = \alpha(g_j(x) - \sum_{i<j} g_i(x))$, where α is as in case 1. For a fixed index k take $g_k(x) = 1$, if $x \in B_{\varepsilon_k}(x_k)$. For all $j > k$, we have $g_j(x) - \sum_{i<j} g_i(x) \leq g_j(x) - 1 \leq 0$. Hence $h_j(x) = 0$, i.e., on the ball $B_{\varepsilon_k}(x_k)$, at most the functions $h_1, \dots, h_k$ are not identically zero. In other words, each point $x \in \mathbb{R}^n$ has a neighborhood (one of the balls $B_{\varepsilon_k}(x_k)$) on which all, but finitely many of the functions h_j, are identically zero.

On the other hand, since each $x \in \mathbb{R}^n$ belongs to a ball $B_{\varepsilon_j}(x_j)$ for some j, then $g_j(x) = 1$ for that j. Let j be minimal such that $g_j(x) > 0$. Therefore $g_i(x) = 0$ if $i < j$. Thus $h_j(x) = \alpha(g_j(x)) > 0$.

If we define $H_\nu : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\nu = 1, 2$ by

$$H_\nu(x) = \sum_{B_{\delta_j}(x_j) \subset V_\nu} h_j(x)\,,$$

then as we saw above, this sum is finite on a neighborhood of x. Hence H_ν is well defined and is smooth. The sum $H_1(x) + H_2(x)$ is clearly positive. If we take $x \in A$, then $x \notin V_1$, and thus $x \notin B_{\delta_j}(x_j)$ if j is such that $B_{\delta_j}(x_j) \subset V_1$. Consequently $g_j(x) - \sum_{i<j} g_i(x) = -\sum_{i<j} g_i(x) \leq 0$ and so

$$H_1(x)\alpha\left(g_j(x) - \sum_{i<j} g_i(x)\right) = 0\,.$$

Moreover, if $x \in B = \mathbb{R}^n - V_2$, then $H_2(x) = 0$.

Now define $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$g(x) = \frac{H_2(x)}{H_1(x) + H_2(x)}\,.$$

Then g is smooth. If $x \in A$, then $g(x) = 1$ and if $x \in B$, then $g(x) = 0$. Finally, for any $x \in \mathbb{R}^n$, one has $0 \leq g(x) \leq 1$, namely, g is as desired. □

Now we already have a smooth version of Theorem 9.5.23, and its proof is virtually the same, up to the fact that instead of Urysohn's lemma 9.1.25, we use its smooth version 9.5.26 that we just proved.

9.5.27 **Theorem**. *Let $\mathcal{C} = \{U_\lambda\}$ be an open cover of $\mathbb{R}^n$. Then there is a smooth partition of unity $\{\pi_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}\}$ subordinate to $\mathcal{C}$.* □

The announced application is the following.

9.5.28 **Theorem**. *Let $X \subset \mathbb{R}^n$ be nonempty, let $f : X \longrightarrow \mathbb{R}$ be continuous and take $\varepsilon > 0$. Then a smooth function $g : X \longrightarrow \mathbb{R}$ exists such that for all $x \in X$, $|g(x) - f(x)| < \varepsilon$.*

Proof: Take $U_x = \{y \in \mathbb{R}^n \mid |f(y) - f(x)| < \varepsilon\}$. The family $\mathcal{C} = \{U_x\}_{x\in\mathbb{R}^n}$ is an open cover. By 9.5.27, there is a smooth partition of unity $\{\pi_x : \mathbb{R}^n \longrightarrow \mathbb{R}\}$ subordinate to $\mathcal{C}$. Let

$$g(y) = \sum_{x\in\mathbb{R}^n} \pi_x(y) f(x)$$

be the assembly of the constant functions with value $f(x)$ defined on each open set U_x. By 9.5.25, g is well defined and smooth. Moreover

$$\begin{aligned} |g(y) - f(y)| &= |\sum_{x\in\mathbb{R}^n} \pi_x(y) f(x) - f(y)| \\ &= |\sum_{x\in\mathbb{R}^n} \pi_x(y)(f(x) - f(y))| \\ &\leq |\sum_{x\in\mathbb{R}^n} \pi_x(y)|f(x) - f(y)| \\ &< \sum_{x\in\mathbb{R}^n} \pi_x(y)\varepsilon \\ &= \varepsilon\,, \end{aligned}$$

since if $\pi_x(y) > 0$, then $y \in U_x$ and therefore $|f(x) - f(y)| < \varepsilon$. □

9.6 INTERRELATIONS AMONG TOPOLOGICAL PROPERTIES

In this last section we shall present a hierarchical diagram with all axioms that a topological space may fulfill. Before doing it, and in order to have a more complete list, we shall give another definition, whose significance we shall not discuss.

9.6.1 DEFINITION. A topological space X is *perfectly normal* if it is Hausdorff and given disjoint closed sets $A, B \subset X$, there is a continuous function

$$u : X \longrightarrow [0, 1]$$

such that $u^{-1}(0) = A$, $u^{-1}(1) = B$.

9.6.2 EXERCISE. Show that every metrizable space X is perfectly normal.

9.6.3 EXERCISE. Let X be a topological space. A set $A \subseteq X$ is called a G_δ*-set* if it is a countable intersection of open sets. Show that a Hausdorff space X is perfectly normal if and only if it is normal and all its closed sets are G_δ sets.

(*Hint:* For sufficiency, if A is closed and $A = \cap G_n$, where each G_n is open, then there is a function u_n such that $u_n|A \equiv 0$ and $u_n|_{X-G_n} \equiv 1$ for all $n \in \mathbb{N}$. Take $u_A(x) = \sum \frac{u_n(x)}{2^n}$. If the sets A and B are disjoint and closed and we take

$$u(x) = \frac{u_A(x)}{u_A(x) + u_B(x)}\,,$$

then u is continuous and $u^{-1}(0) = A$, $u^{-1}(1) = B$.)

9.6.4 Exercise. Show that every perfectly normal space X is completely normal.

9.6.5 Exercise. Let X be a metric space with metric d. If A and B are nonempty compact sets in X, define their distance by

$$\widehat{d}(A, B) = \max(\max_{x \in A} d(x, B), \max_{y \in B} d(y, A))\,.$$

Show that $\widehat{d}$ is a metric, which is known as *Hausdorff metric* on the set of nonempty compact sets in X (cf. 1.3.15).

9.6.6 Exercise. Let X be a locally compact Hausdorff space. Prove that the following are equivalent:

(a) X is second-countable.

(b) The Alexandroff construction X^* is metrizable.

(c) X es metrizable and countable at ∞.

9.6.7 Exercise. Prove that a topological manifold is metrizable and countable at ∞.

We can summarize the interrelations of possible properties of topological spaces in the diagram pictured in Figure 9.1.

9.6.8 Exercise. Consider all implications in the diagram of Figure 9.1 and prove those which were not proved in the text.

9.6.9 Examples. Consider the following list of topological spaces:

(1) X = indiscrete space with more than one point.

(2) The Sorgenfrey line.

(3) The Sierpinski space.

(4) $\mathbb{Q}$ with the relative topology induced by usual topology of $\mathbb{R}$.

(5) $\mathbb{R} - \mathbb{Q}$ with the relative topology induced by usual topology of $\mathbb{R}$.

(6) $[0, \Omega)$, where Ω is the first uncountable ordinal, with the order topology.

(7) $\mathbb{R}^n$.

(8) $\mathbb{R}^\infty$.

(9) $\mathbb{R}^\omega$.

(10) X as in 9.5.17.

(11) $Y = X \times X$, if X is as in (7).

(12) The Hilbert-cube I^ω.

(13) $X = [-1, 1]$ with the topology $\mathcal{A} = \{A \mid 0 \notin A \text{ or } (-1, 1) \subset A\}$.

(14) X uncountable, with the topology $\mathcal{A} = \{A \mid X - A \text{ is finite or } p \in X - A\}$, where $p \in X$ is a particular point.

9.6.10 Exercise. In the diagram of Figure 9.1, next page, there are implications which are given only in one direction. From the examples above, choose those which are counterexamples for the other direction.

9.6.11 Note. The book of Steen and Seebach [17] is a natural source of additional counterexamples to those of the list. The reader should look at it for more elaborate examples to see in some cases that the implications in the diagram are valid only in one direction.

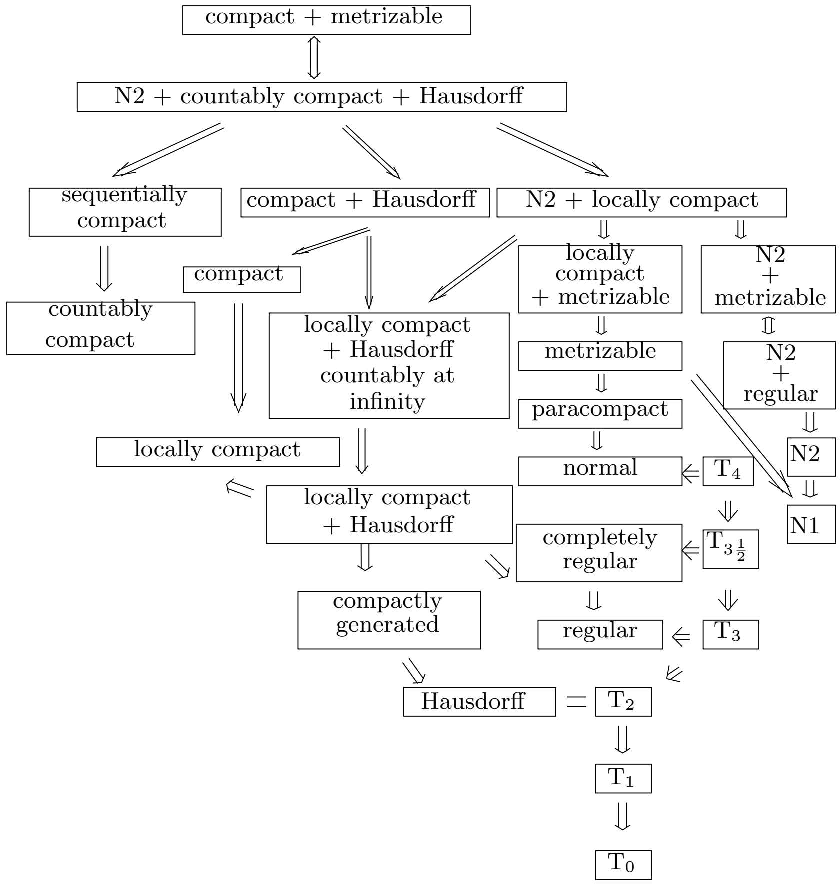


Figure 9.1 Interrelations among topological properties

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Alphabetical index

SYMBOLS